Group 9
Entropy and Area

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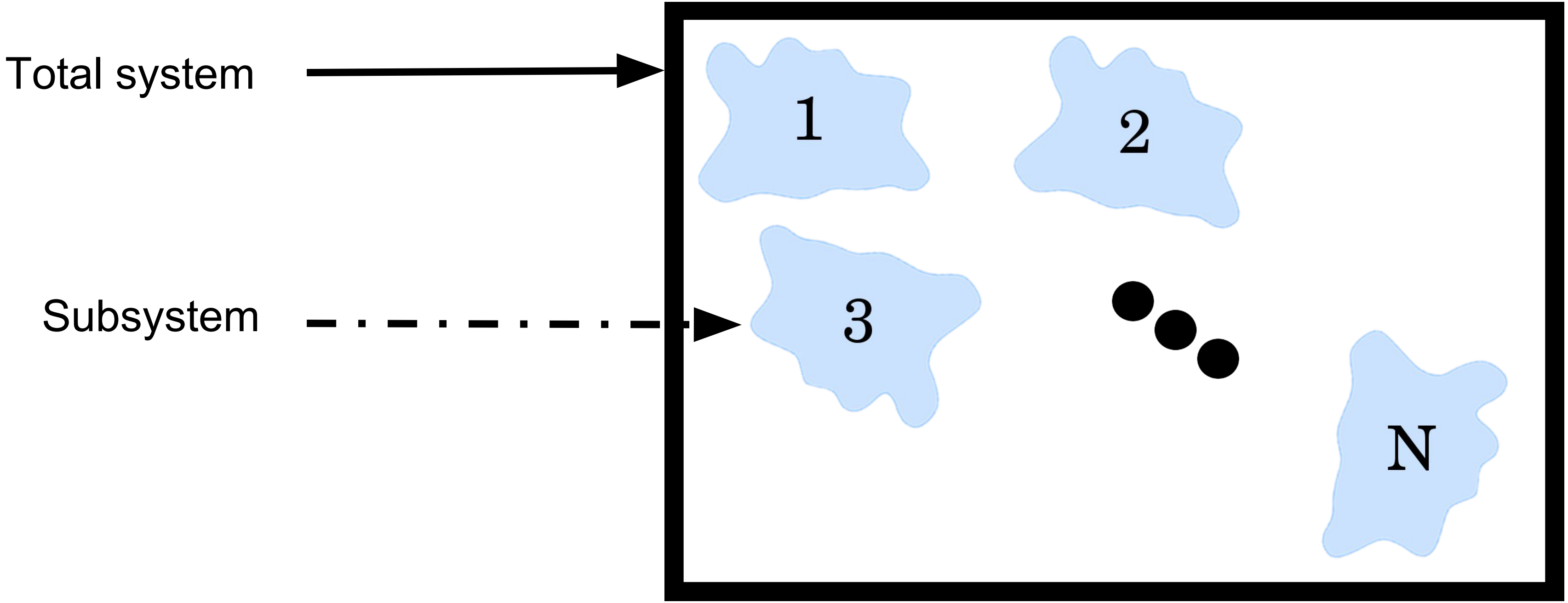
Srednicki, M.
Entropy and area
(1993) Physical Review Letters, 71 (5), pp. 666-669.

A system can be Partitioned

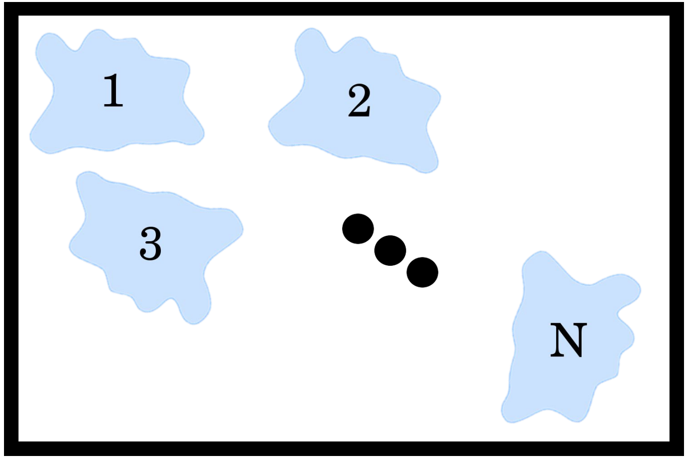
Consider an arbitrary system; one can mentally (or physically) partition the total system into N subsystems.



Entanglement does not emerge in Classical Mechanics

The system is described by specifying the properties of each subsystem:

$$(X, P) = (x_1, p_1 | x_2, p_2 | \cdots | x_N, p_N)$$

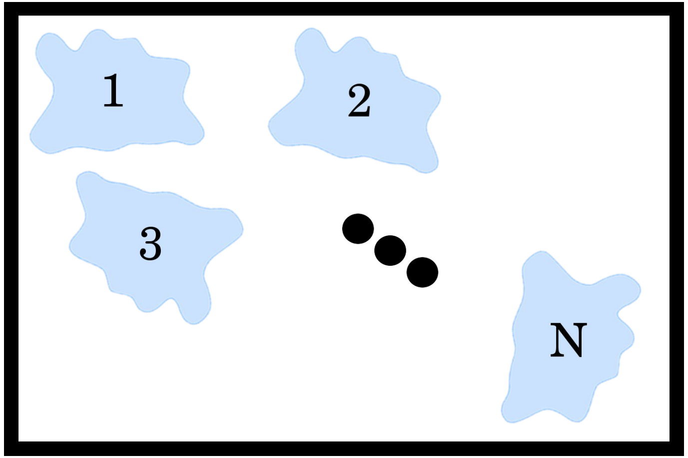


If one knows the position and momentum of every particle, one knows everything about the system.

Entanglement emerges from superposition in Quantum Mechanics

The system is described by a *superposition* of states of each subsystem:

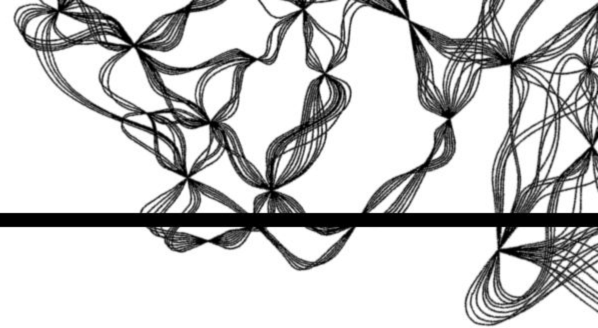
$$|\Psi\rangle = \sum_{n_1,n_2,\ldots,n_N} M_{n_1,n_2,\ldots,n_N} |\psi^1_{n_1}\rangle \otimes |\psi^2_{n_2}\rangle \otimes \cdots \otimes |\psi^N_{n_N}\rangle$$



In general, the state of a system cannot be written as the product of subsystem states:

- States can be **entangled**.

Example: Entanglement with two Spin—1/2 particles



$$|\Psi\rangle = a\,|\uparrow\rangle\,|\uparrow\rangle + b\,|\uparrow\rangle\,|\downarrow\rangle + c\,|\downarrow\rangle\,|\uparrow\rangle + d\,|\downarrow\rangle\,|\downarrow\rangle$$

Product States:

$$|\Psi\rangle = |\uparrow\rangle\,|\uparrow\rangle$$

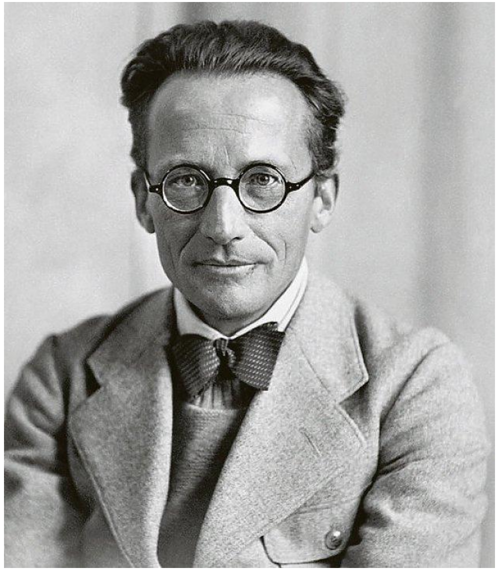
$$\begin{aligned} |\Psi\rangle &= a\,|\uparrow\rangle\,|\uparrow\rangle + b\,|\uparrow\rangle\,|\downarrow\rangle \\ &= |\uparrow\rangle\,(a\,|\uparrow\rangle + b\,|\downarrow\rangle) \end{aligned}$$

Entangled States:

$$|\Psi\rangle = a\,|\uparrow\rangle\,|\uparrow\rangle + d\,|\downarrow\rangle\,|\downarrow\rangle$$

$$\begin{aligned} |\Psi\rangle &= a\,|\uparrow\rangle\,|\uparrow\rangle + b\,|\uparrow\rangle\,|\downarrow\rangle + c\,|\downarrow\rangle\,|\uparrow\rangle \\ &= |\uparrow\rangle\,(a\,|\uparrow\rangle + b\,|\downarrow\rangle) + c\,|\downarrow\rangle\,|\uparrow\rangle \end{aligned}$$

Schrödinger's "Verschränkung"



Schrödinger coined the term entanglement.

Though we can describe the entangled state as a whole, we cannot assign individual states to the subsystems:

- Maximal information about the total system.
- Minimal information about the subsystems.

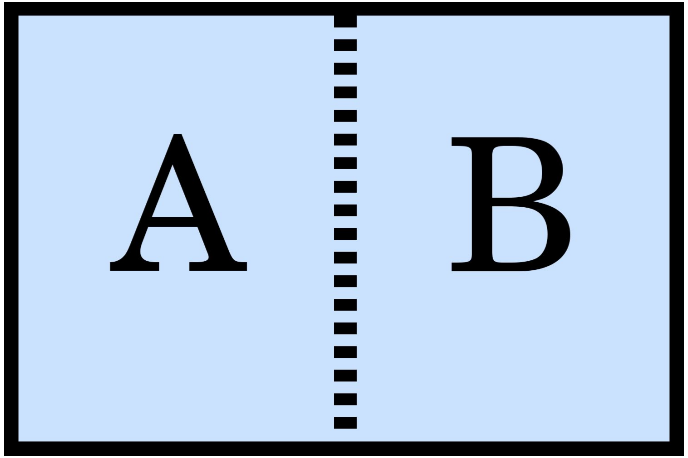
"The best possible knowledge of a whole does not include the best possible knowledge of its parts — and this is what keeps coming back to haunt us."*

* Schrödinger E, 1935, Naturwissenschaften 23, 807.

To make things concrete, consider Bipartite Entanglement

Consider a system partitioned into 2 subsystems, A and B.

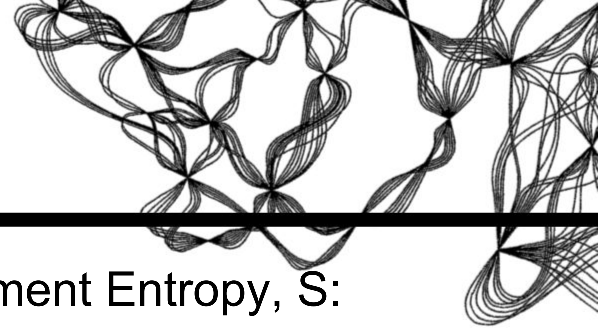
$$|\Psi\rangle = \sum_{a,b} M_{a,b} \, |\psi_a^A\rangle \otimes |\psi_b^B\rangle$$



The Schmidt Decomposition:

$$|\Psi\rangle = \sum_{n} \sigma_n \, |\psi_n^A\rangle \otimes |\psi_n^B\rangle$$

The Entanglement Entropy quantifies the degree of Entanglement



Determine the ‘degree of entanglement’ with the Entanglement Entropy, S:

$$S = -\sum_n \sigma_n^2 \ln \sigma_n^2$$

Product States:

$$|\Psi\rangle = |\uparrow\rangle\,|\uparrow\rangle$$

$$S = 0$$

Entangled States:

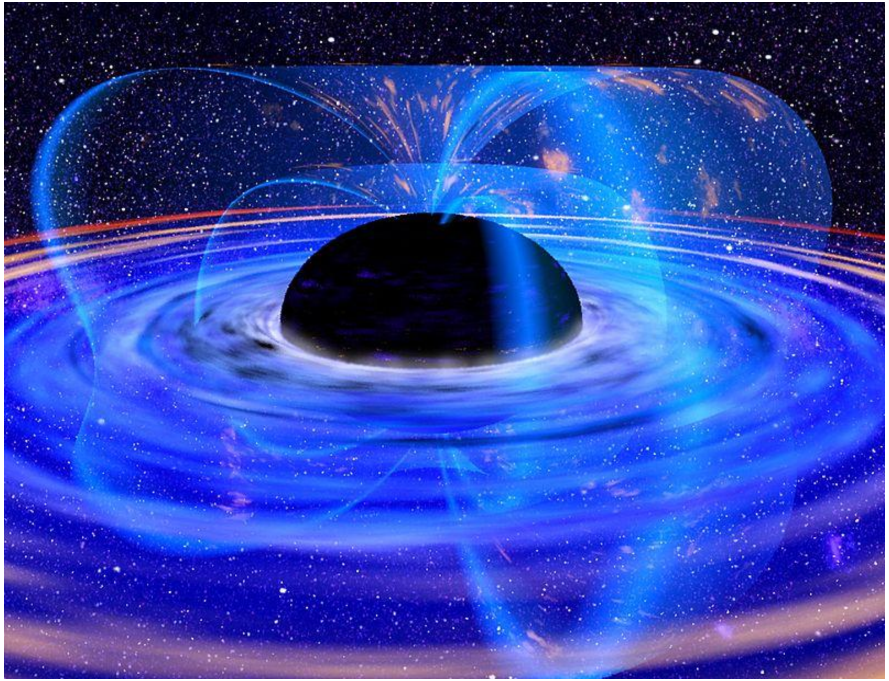
$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle\,|\downarrow\rangle - |\downarrow\rangle\,|\uparrow\rangle)$$

$$S = \ln(2)$$

The Entanglement Entropy follows from the Reduced Density Matrix:

$$\rho_A = \mathrm{Tr}_{\mathrm{B}}[|\psi\rangle\,\langle\psi|] = \sum_n \sigma_n^2\,|\psi_n^A\rangle\,\langle\psi_n^A| \rightarrow S = -\mathrm{Tr}[\rho_A \ln \rho_A]$$

Black Holes obey an Area Law



Hawking showed the entropy of a black hole scales with area of its event horizon.

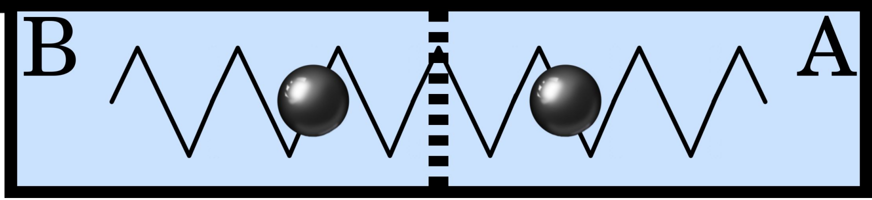
$$S \sim A$$

Srednicki discovered this is generic — The ground states of most quantum systems obey an **Area Law**:

- Entropy scales with the area of the boundary between two subsystems.

First System: 2 Coupled Oscillators

$$H = \frac{1}{2}\left[p_1^2 + p_2^2 + k_0(x_1^2 + x_2^2) + k_1(x_1 - x_2)^2\right]$$



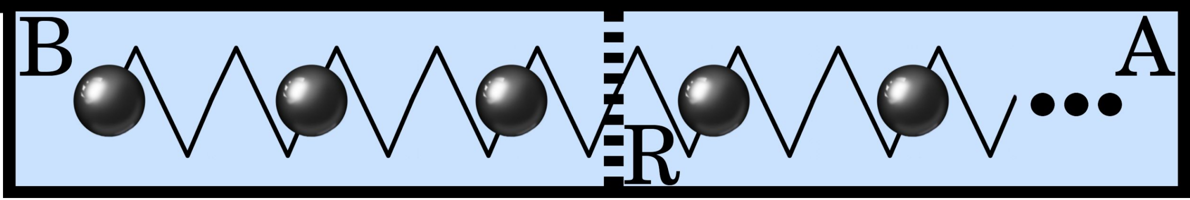
Find the ground state Ψ_0 and trace out the first oscillator:

$$\rho_A(x_2, x_2') = \int_{-\infty}^{\infty} dx_1 \psi_0(x_1, x_2)\psi_0^*(x_1, x_2')$$

Diagonalize the Reduced Density Matrix: $\sigma_n^2 = (1-\lambda)\lambda^n \quad , \lambda = \lambda(k_0, k_1)$

$$S = S(\lambda)$$

Next System: N Coupled Oscillators



N harmonic oscillators characterized by a matrix K.

$$H = \frac{1}{2}\sum_{i=1}^{N} p_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i K_{i,j} x_j$$

Trace out the inner n oscillators.

- Analogous to a black hole!

The entropy of this partition has the form:

$$S = \sum_i S(\lambda_i)$$

The λ_i are related to the eigenvalues of a matrix built around K.

Final System: Infinite Oscillators

Free massless scalar field: $H = \frac{1}{2}\int d^3x \left[\pi^2 + (\nabla\varphi)^2\right]$

Expand the field operators in order to separate the Hamiltonian: $H = \sum_{l,m} H_{l,m}$

$$H_{l,m} = \frac{1}{2a}\sum_{j=1}^{N}\left\{\pi_{l,m}^2(j) + (j+1/2)^2\left[\frac{\varphi_{l,m}(j+1)}{j+1} - \frac{\varphi_{l,m}(j)}{j}\right]^2 + \frac{l(l+1)}{j^2}\varphi_{l,m}^2(j)\right\}$$

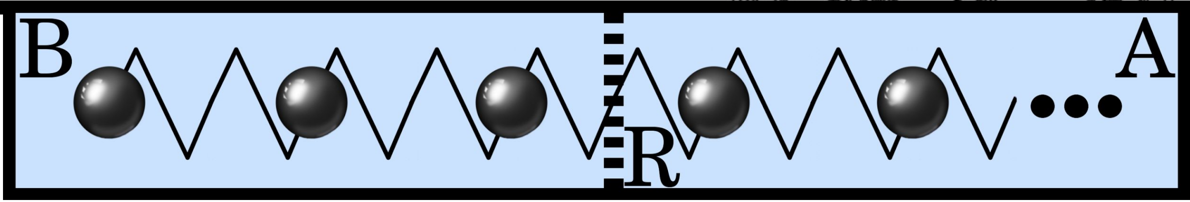
Same form as N coupled oscillators!

$$S = \sum_{l,m} S_{l,m}$$

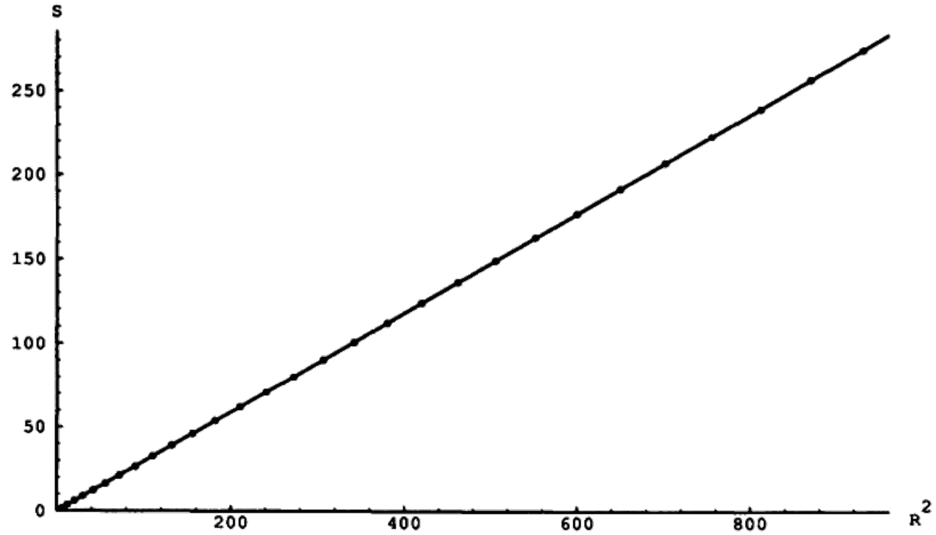
* $S_{l,m}$ = Entropy for N coupled oscillators.

Entropy scales with the Radius of the Boundary Squared

Numerics reveal that the system obeys an Area law.



$$R = (n + 1/2)a$$



$$S = \kappa M^2 R^2$$

κ (=0.30) is a proportionality constant.

M is the inverse lattice spacing.

* Srednicki M, 1993, Phys. Rev. Lett. 71, 666.

“Area Laws” are also present in other dimensions

Number of Dimensions	Entropy
2	S ~ R
1	S ~ ln(R)
≥4	Partial sums do not converge

Critical Analysis

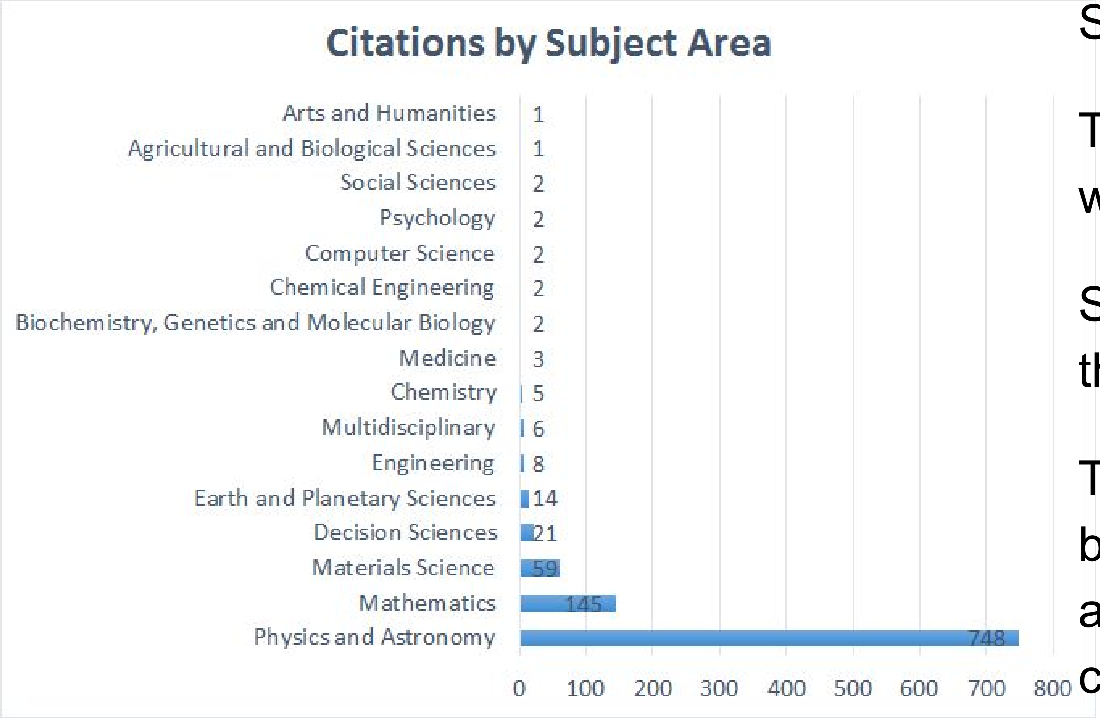
Improvements:

Several heuristic arguments are given throughout the paper.

Author argues that it should be reasonable to expect the entropy to scale with R^2 rather than R^3 without any rigorous treatment, his argument relies largely on intuition.

An explicit equation of the form $S = \kappa M^2 R^2$ is mentioned but never formally derived. This is also true for the n - dimensional cases.

Critical Analysis



Successes:

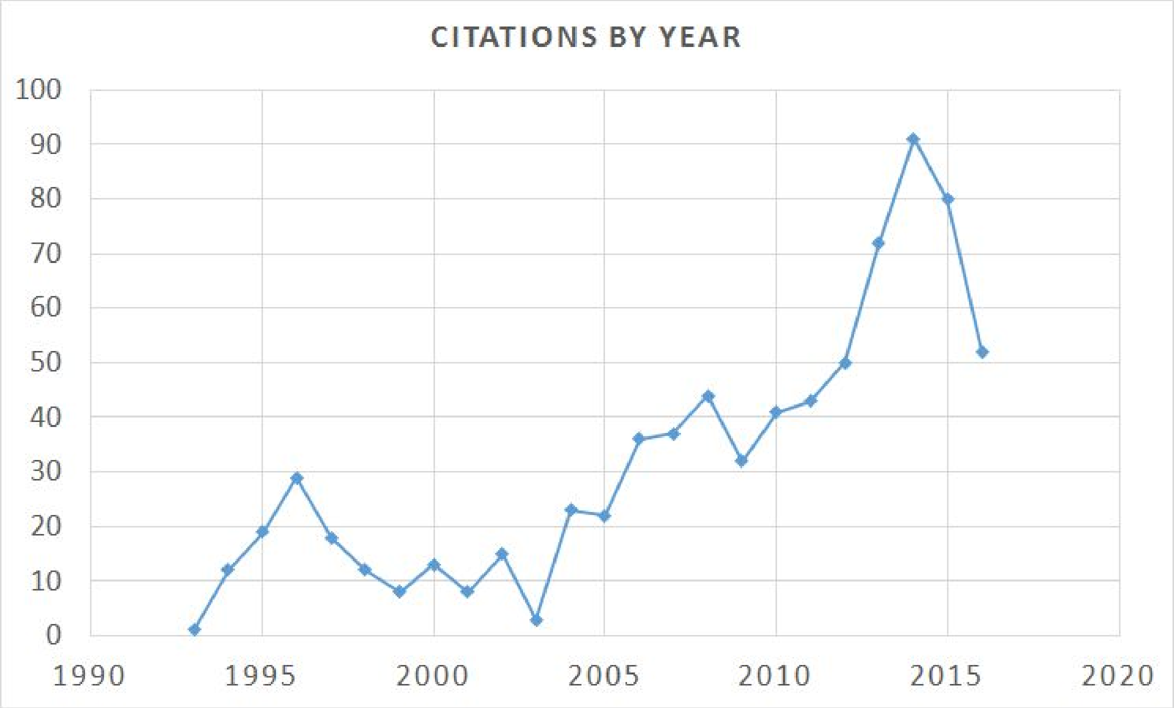
The content is accessible to readers within diverse scientific backgrounds.

Several examples are presented through the discussion.

The paper has a logical structure. It begins by considering simple systems and progressively moves into more complex examples.

Citations

Srednicki's paper was published in Physical Reviews Letters on August 2nd, 1993. Since then, it has been cited more than 760 times.



Evolution of field: Haldane's Entanglement Hamiltonian

The entanglement entropy follows from the Reduced Density Matrix.

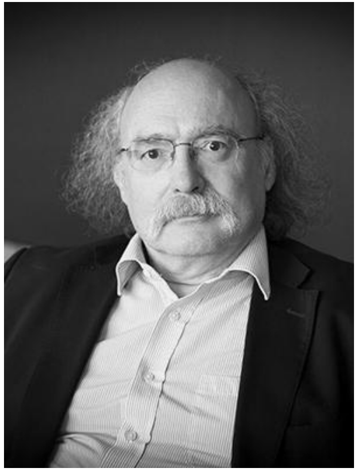
Haldane showed us:*

$$\rho_A = \frac{1}{\mathcal{Z}} e^{-\mathcal{H}_A}$$

The log of the Schmidt eigenvalues behave as the eigenvalues of an **Entanglement Hamiltonian**.

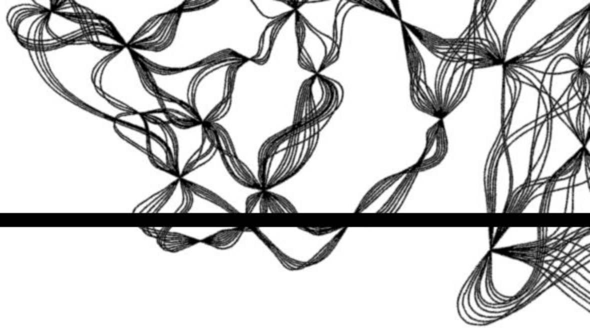
Event horizon = Physical boundary.

Entanglement cut = Mental boundary.



* Haldane F D M and Li H, 2008, Phys. Rev. Lett. 101, 010504.

Thank you!



References

[1] Schrödinger E, 1935, Naturwissenschaften 23, 807.

[2] Hawking S W, 1975, Comm. Math. Phys. 43, no. 3, 199.

[3] Srednicki M, 1993, Phys. Rev. Lett. 71, 666.

[4] Haldane F D M and Li H, 2008, Phys. Rev. Lett. 101, 010504.

[5] Amico L, Fazio R, Osterloh A and Vedral V, 2008, Rev. Mod. Phys. 80, 517.

[6] Peschel I, 2012, Braz. J. Phys. 42, 267.

Image credit: Eliza Stamps, Old Man in the Mountain