UNIT 12: TWO-STATE SYSTEMS

After this unit, you should be able to

- Predict the probability of a photon passing through a sequence of polarizing filters, given its initial polarization state.

- Given a spin wave function, compute the probability of transmission through a Stern-Gerlach device.

Light polarization as a quantum state

Light can be polarized either horizontally or vertically. How do we describe that for a photon using quantum mechanics? Let's suppose that the photon is moving in the z direction, so its polarization is in the x, y plane. Let's call horizontal (x) polarization Ψ_h and vertical (y) polarization Ψ_v. The quantum state associated with each of them we will call h and v. A general quantum state for polarization will be $\Psi = a\Psi_h + b\Psi_v$, where a and b can be any complex numbers, so long as $|a|^2 + |b|^2 = 1$. Note that in this case, we have only discrete possibilities, as opposed to the case when we are considering the probability that a particle is at a given position ($\Psi(x)$ from the previous section*).

In quantum mechanics, diagonal and circular polarization are written as *superpositions* of vertical and horizontal polarization as shown in Table 1. Classically, you might expect there to be a continuum of values. Why don't we write it as $\Psi(\theta)$, with θ the angle from the y axis? This is actually a very deep question which we cannot answer very rigorously in this class; you will learn this in advanced quantum mechanics. One way of looking at this is to note that in polarization, a diagonal polarization is $\Psi_h + \Psi_v$. On the other hand, a particle at $x = 1$ nm is not a superposition of the particle being at $x = 0$ nm and $x = 2$ nm. For this course, it suffices to know that some things (position, momentum) are represented by a continuous variable while other things (polarization) are represented by discrete variables.

Table 1: Different polarizations of light written as linear combinations of horizontal(h) and vertical(v) polarization.

Polarization direction	State
Vertical	Ψ_v
Horizontal	Ψ_h
Diagonal (45 degrees)	$\frac{1}{\sqrt{2}}(\Psi_h + \Psi_v)$
Diagonal (-45 degrees)	$\frac{1}{\sqrt{2}}(\Psi_h - \Psi_v)$
Circular (right-handed)	$\frac{1}{\sqrt{2}}(\Psi_h + i\Psi_v)$
Circular (left-handed)	$\frac{1}{\sqrt{2}}(\Psi_h - i\Psi_v)$

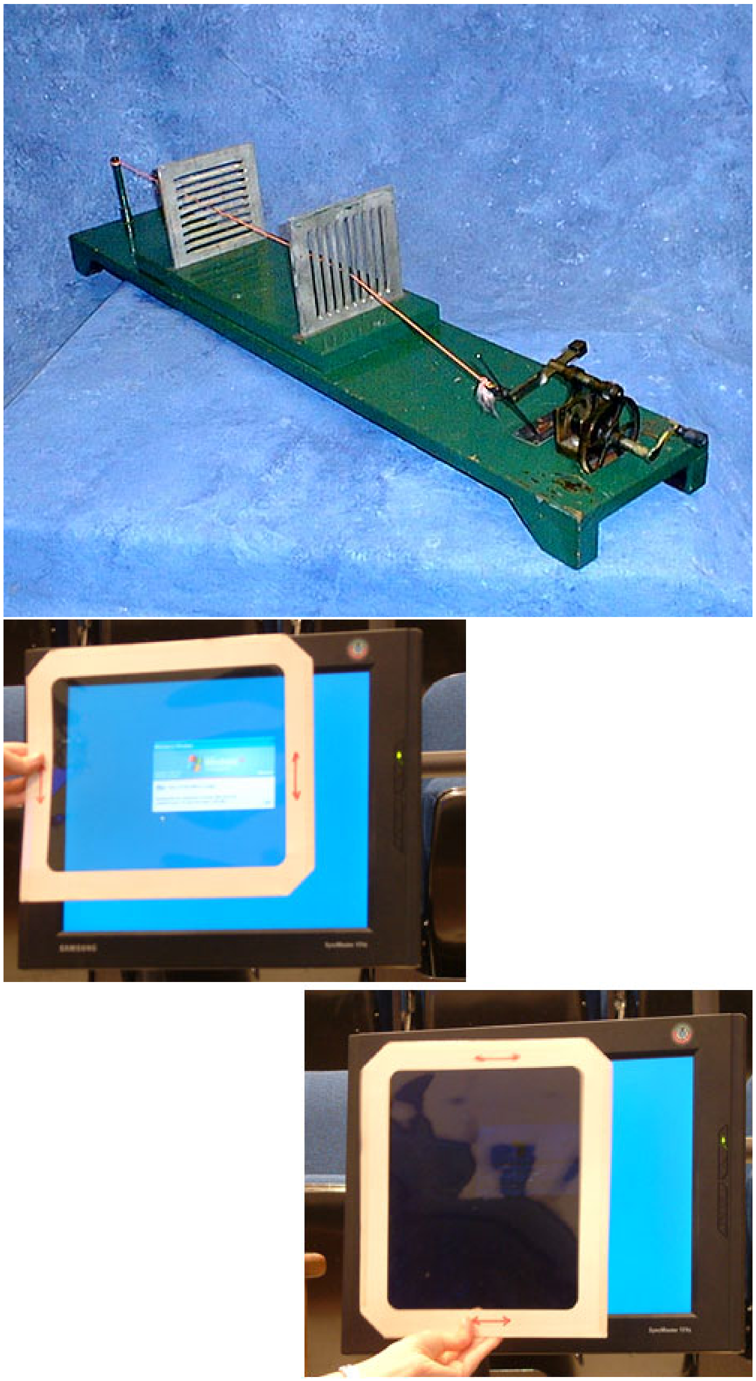

Figure 1: (top) A model of how polarization filters work. If the lines are arranged in opposite ways, the electric field cannot oscillate in either direction, and the light is blocked. (bottom) Light from LCD screens is polarized. If the filter is held one way, you can see the screen, if it's held the other way, it's blocked.

Table 2: States with definite spin in different directions for systems with two spin possibilities.

Spin direction	State
$\hat{z}$	$\uparrow$
$-\hat{z}$	$\downarrow$
$\hat{x}$	$\frac{1}{\sqrt{2}}(\uparrow + \downarrow)$
$-\hat{x}$	$\frac{1}{\sqrt{2}}(\uparrow - \downarrow)$
$\hat{y}$	$\frac{1}{\sqrt{2}}(\uparrow + i \downarrow)$
$-\hat{y}$	$\frac{1}{\sqrt{2}}(\uparrow - i \downarrow)$

Measurement of polarization

Now let's suppose what happens when a photon encounters a filter that only lets vertically polarized light through it. What do you suppose will happen? Surely, if the photon is vertically polarized, it will simply pass through the filter. On the other hand, if the light is horizontally polarized, it will not pass through the filter.

What about when the light has a diagonal polarization, so that its polarization is given by $\frac{1}{\sqrt{2}}(\Psi_h + \Psi_v)$? It turns out that the photon has a **50% probability** of passing through a vertical filter. And after the photon passes through the filter, its polarization is Ψ_v. Why has it changed? Well, we said before that only vertically polarized light passes through the filter, so if the photon passed through the filter, it must be vertically polarized.

The polarization Ψ is telling us the *probability* that the photon will pass through a filter. If the polarization is given by $a\Psi_v + b\Psi_h$ (with $|a|^2 + |b|^2 = 1$), then the probability of the photon passing through the vertical filter is given by $|a|^2$, and of passing through the horizontal filter is given by $|b|^2$.

Electron and atom spin

The spin of an electron or some atoms is described very similarly to the polarization of light (remember, matter and light are very similar in quantum mechanics!). In these systems, there is an internal quantity called *spin*[1] that can either point up or down (or left or right, or forward or backward). This results in a magnetic moment either pointing up or down. This applies to several different types of systems: electrons, neutrons, protons, deuterium (hydrogen with a proton and a neutron), silver atoms (Ag), and other atoms. The states are given as in Table 2, where $\uparrow$ means that the magnetic moment is pointed in the $+\hat{z}$ direction and $\downarrow$ means that the magnetic moment is pointed in the $-\hat{z}$ direction.

[1] It is called spin because the particle acts as if it's spinning, which gives rise to angular momentum and a magnetic field. However, this is angular momentum *without* classical rotation. Yep.

Figure 2: Stern-Gerlach experimental setup. The changing magnetic field separates randomly oriented atoms into two streams.

Measurement of spin and a generalized rule for measurement

Let's suppose that we have an arbitrary spin state given by $a \uparrow + b \downarrow$. Then we can compute the probability of measuring a spin pointing up by using the same rule as we had before:

$$P(\hat{z}) = \frac{|a|^2}{|a|^2 + |b|^2}, \tag{1}$$

and a similar rule for $P(\hat{z})$. Remember the collapse rule: if we do measure spin in the $\hat{z}$ direction, then afterwards the wave function is $\uparrow$, even if it was something else before.

To find the probability of observing the spin in the $\hat{x}$ direction, we have to use a slightly more generalized version of the same rule, which is very similar to the dot product you may have seen in vector math. Assume that $|a|^2 + |b|^2 = 1$; then,

$$P(\hat{x}) = \left| \frac{1}{\sqrt{2}} (\uparrow + \downarrow)^* \cdot (a \uparrow + b \downarrow) \right|^2 = \left| \frac{1}{\sqrt{2}} (a + b) \right|^2. \tag{2}$$

Note the complex conjugate for the state that we're measuring. Similarly,

$$P(\hat{y}) = \left| \frac{1}{\sqrt{2}} (\uparrow + i \downarrow)^* \cdot (a \uparrow + b \downarrow) \right|^2 = \left| \frac{1}{\sqrt{2}} (a - ib) \right|^2. \tag{3}$$

You can verify for yourself that this rule has some sensible properties: a state with definite spin direction will, with probability 1, be observed in that direction. In general, the measurement rule for a particle with wave function Ψ and a state with definite direction S is: $P(S) = |S^* \cdot \Psi|^2$. Make sure to normalize your wave function to use this rule. You can also use this rule for polarization! A strange fact of this rule is that if the spin is definite in $\hat{z}$, then it is maximally indefinite in $\hat{x}$ and $\hat{y}$; a measurement in each of these directions has a 50/50 chance of being positive or negative.

Stern-Gerlach experiment

The Stern-Gerlach experiment is one of the experiments that really show that we have to use the description of spin. The rules presented in this chapter are the simplest ones that have been come up with that also predict the output of this experiment. A measurement of the spin direction is performed by sending the particle through a changing magnetic field. The particle will be deflected an amount proportional to the alignment of the particle's magnetic moment with the magnetic field. Classically, one would expect to see a range of deflections, depending on which direction the magnetic moment happened to be pointing. However, in reality, we only see **two** deflections; one up and one down. This already is evidence for the quantum nature of spins; when we measure the spin, we only get one of two values.

Things get very interesting when we perform multiple experiments in a row on the same atoms. Let's consider adding a second measurement in the $\hat{z}$ direction after the one in Fig 2, but so that it only intercepts the $+z$ atoms. Since we already measured the z direction and found it to be $+z$, the wave function is simply $\uparrow$. With probability 1, the atoms pass through the upper path. This is similar to the polarization behavior!

Now suppose we measure in the x direction and choose the $+x$ atoms. Their wave function is now $\frac{1}{\sqrt{2}}(\uparrow + \downarrow)$. If we then measure the z direction, then the atoms will go up and down with equal probability, since it is an equal superposition of $\uparrow$ and $\downarrow$. And so on, there are many fun games one can play with this. The behavior of spins is another instance of an uncertainty principle, similar to what we saw with momentum and position. If the spin direction in the z-axis is definite, then the x and y directions are uncertain, and vice versa.

Philosophical interlude

This is again a place where you might reasonably ask why is it this way, so that the x-direction of spin can never be definite when the y direction of spin is definite. A similar motif appears in the uncertainty principle between momentum and position. There are actually several levels of understanding this. The first, and most important is that it works. This math predicts the outcome extremely precisely not just the experiments we have presented here, but also a number of other, even stranger experiments in which multiple spins interact with each other. At its core, physics is about making precise models of reality and so if our mathematical model represents the experiments accurately, then to some extent we have done our job, and this does it.

One might also ask whether "this is it;" that is, are there deeper principles that have this behavior as a result? The answer is yes, there are, although they are a bit beyond this class. Actually there are a number of principles that you've already seen which are the result of deeper principles, for example, the constant speed of light(special relativity, which is in reach for many of you), conservation of energy (Noether's theorem, which takes a little while), and spin (the combination of special relativity and quantum mechanics). Keep asking why, and you'll eventually get to an unknown, though. You can take the principles we've learned in this class and perfectly successfully apply them to many physical problems of relevance, or you can choose to explore the deeper principles. Either path is valid!