Unit 2: Interference

After this unit, you should be able to

- Compute the phase difference at an observer's position between the waves emanated from two sources, then compute the intensity that the observer experiences.
- Apply interference rules to two-slit, interferometer, and other interference problems where the path length and source phases differ.

Superposition of waves

In the previous section*, we considered a single harmonic wave generated by a source. Now suppose that there are two sources (1 and 2) generating two waves. For this class, for simplicity, we will only consider waves with the same wavelength, amplitude, and speed, so they have the same A, ω, and k:

$$y_1(x,t) = A\cos(kx - \omega t + \phi_1) \tag{1}$$
$$y_2(x,t) = A\cos(kx - \omega t + \phi_2). \tag{2}$$

In many cases (you can always assume this in this course), the total wave is given by superposition [1], so

$$y_{\text{total}}(x,t) = y_1(x,t) + y_2(x,t). \tag{3}$$

At some times and positions, y_1 and y_2 might be either the same sign or different signs. So at some locations the summed amplitude will be larger than either of the waves by themselves, and at other locations it will be smaller. We refer to this as **interference**. If the waves are the same sign, they will interfere *constructively* and if they are opposite signs, they will interfere *destructively*. A picture of this is shown in Figure 1.

Using the trigonometric identity

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \tag{4}$$

we can find that

$$y_{\text{total}}(x,t) = 2A\cos\left(\frac{\phi_1-\phi_2}{2}\right)\cos\left(kx - \omega t + \frac{\phi_1+\phi_2}{2}\right) \tag{5}$$

So the result is a new cos wave with a new amplitude related to the difference in phase, $2A\cos\left(\frac{\phi_1-\phi_2}{2}\right)$. That means that the average intensity is given by $2A^2\cos^2\left(\frac{\phi_1-\phi_2}{2}\right)$, so the difference in **phase** between the waves is the key to knowing the total intensity. There are a few ways of getting differences in the phase.

[1] Waves that do this are called "linear" because they add.

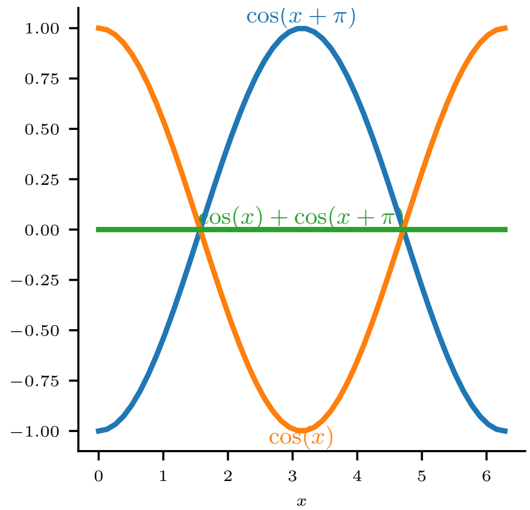

Figure 1: Destructive interference between two waves

Phasors

What if the amplitudes of the waves are different? In that case, we cannot use the trigonometric identity in Eqn 4. Luckily, there is a graphical way of adding harmonic waves, called phasors.

An observer at a given position will see an amplitude from a given source that varies like $A_1 \cos(\phi_1 - \omega t)$. Phasors map this onto a two dimensional vector (Figure 2) with amplitude A_1 and an angle from the x-axis given by $\phi_1 - \omega t$. Then the amplitude at any given time is given just by the x coordinate of the vector, by trigonometry. It turns out that you can add phasors as if they are vectors, and the x coordinate of the summed vector will give the amplitude of the wave at a given time. The length of the phasor squared, divided by two, will give the average intensity of the summed wave.

Example: Two speakers

Now let's consider the intensity of waves emitted by two sources, as measured by an observer. We will consider an observer that is r_1 away from source 1 and r_2 away from source 2 (Figure 3) For simplicity, let's suppose that the sources have exactly the same amplitude. For example, this could be a person listening to music from two speakers. Our objective will be to compute how loud the observer percieves the sound to be. The waves that the observer experiences[2] from each source are:

$$y_1(x,t) = A\cos(kr_1 - \omega t + \phi_1) \tag{6}$$
$$y_2(x,t) = A\cos(kr_2 - \omega t + \phi_2). \tag{7}$$

[2]This is the amplitude *at the observer's position*.

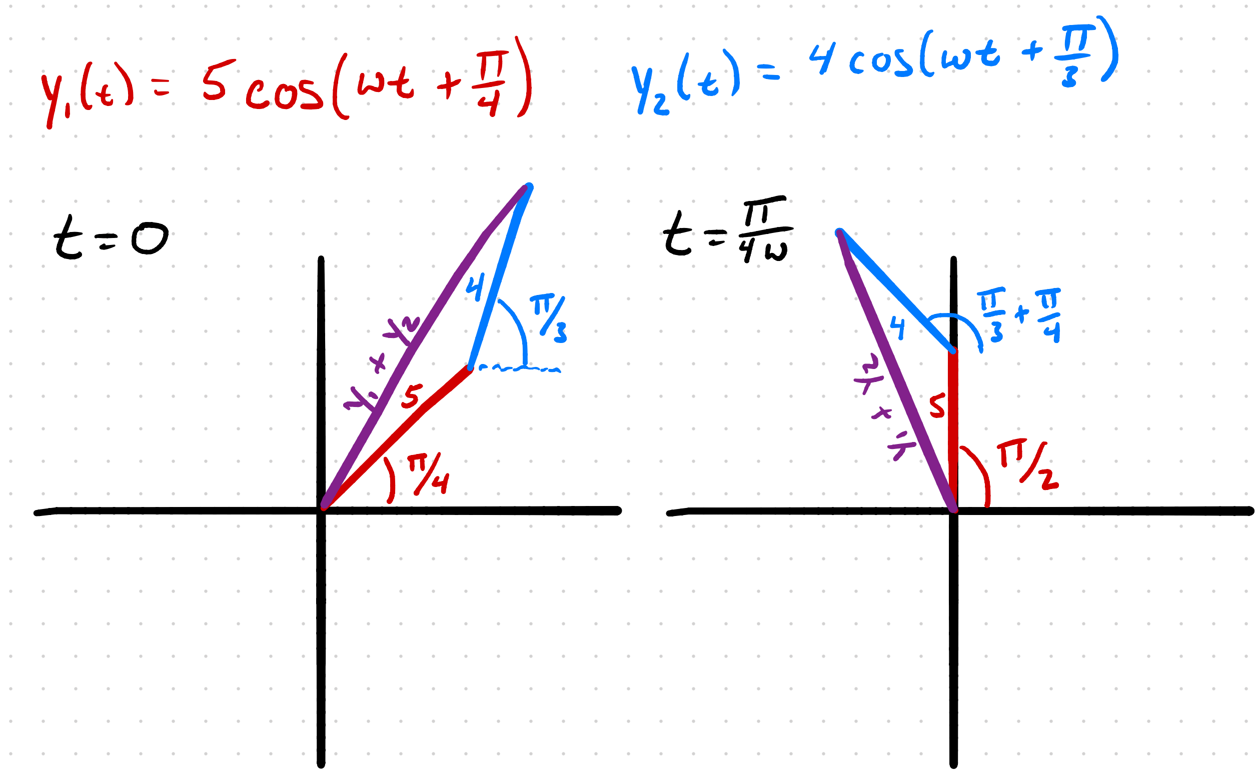

Figure 2: A figure that shows the relationship between phasors and harmonic waves.

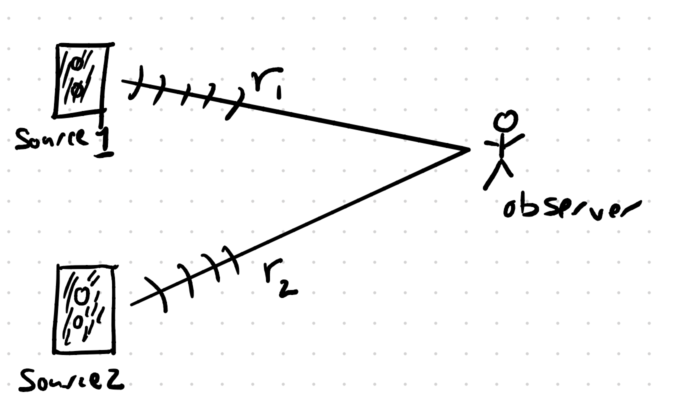

Figure 3: An observer experiencing the intensity of waves generated from two sources in phase.

Then, using the trigonometric identity from Eqn 4,

$$y_1(x,t) + y_2(x,t) = 2A\cos\left(\frac{kr_1 + \phi_1 - kr_2 - \phi_2}{2}\right)\cos\left(\frac{kr_1 + \phi_1 + kr_2 + \phi_2}{2} - \omega t\right). \tag{8}$$

and the total average intensity for that observer is

$$I_{\text{total}} = 2A^2\cos^2\left(\frac{kr_1 + \phi_1 - kr_2 - \phi_2}{2}\right). \tag{9}$$

So the measured intensity depends on the relative phase offsets of the speakers (ϕ_1, ϕ_2) as they are emitting the sound and the position of the observer relative to the two speakers.

Let's suppose that $\phi_1 = \phi_2$ (when this is true, we say that the sources are in phase), so that

$$I_{\text{total}} = 2A^2\cos^2\left(k\frac{r_1 - r_2}{2}\right) = 4I_1\cos^2\left(k\frac{r_1 - r_2}{2}\right). \tag{10}$$

The intensity is maximal ($2A^2$) when $r_1 - r_2 = m\lambda$, where m is an integer, and it is zero when $r_1 - r_2 = \left(m + \frac{1}{2}\right)\lambda$ for some (possibly other) integer m.

Because amplitudes add, and intensity is amplitude squared, the maximal intensity is actually **four** times as large as the intensity of a single source. Energy is still conserved here, because there are places with zero intensity. Interference is just redistributing the total energy compared to the two sources operating independently.

Example: Two slits

An important example of interference is the two-slit experiment, shown in Fig 4. In this experiment, we aim a laser at an opaque barrier with two slits made in it. There is a screen placed a distance L away from the barrier. We measure the intensity of light on the screen as a function of the position on the screen.

We use the physics of interference to analyze this situation.

- We treat the slits as if they are sources of waves.
- Since both slits are equidistant from the laser, the waves at each slit are in phase; that is, $\phi_1 = \phi_2$ when considered at the slit.
- The variation of intensity measured on the screen will be due to the fact that different points on the screen are different distances from the slits.
- If the slits are the same size, then the amplitude measured at the screen will be the same from each slit ($A_1 = A_2$).
- The wavelength coming from both slits is the same, since the same laser is incident ($k_1 = k_2$ and $\omega_1 = \omega_2$)

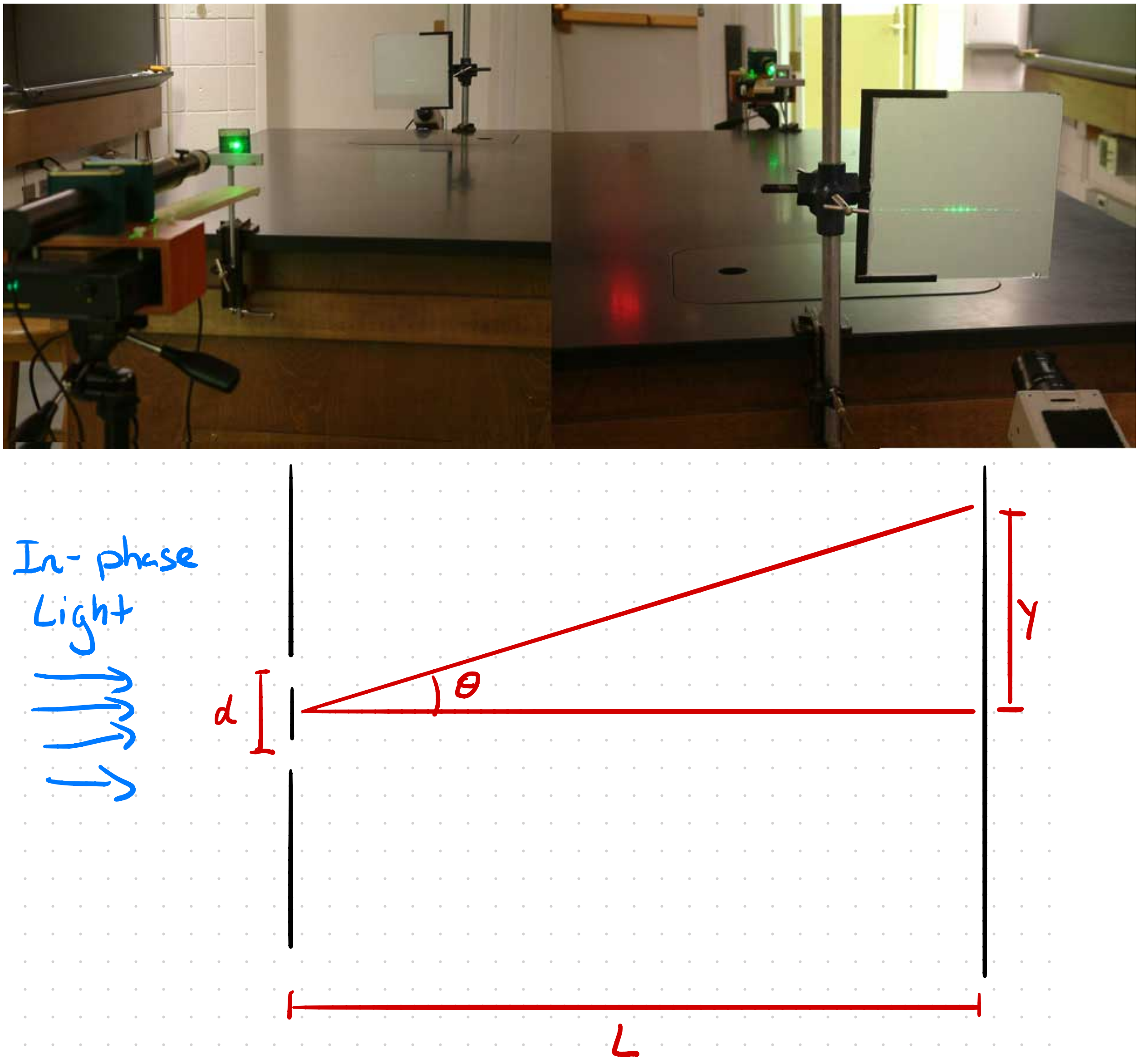

Figure 4: Geometric setup of a two-slit experiment. The card in the pictures has slits separated by a distance d (usually a few μm). The screen is placed L (a meter or two) away from the card. Then the intensity varies on the screen as we can see from a pattern of bright and dark spots. The position on the screen is y.

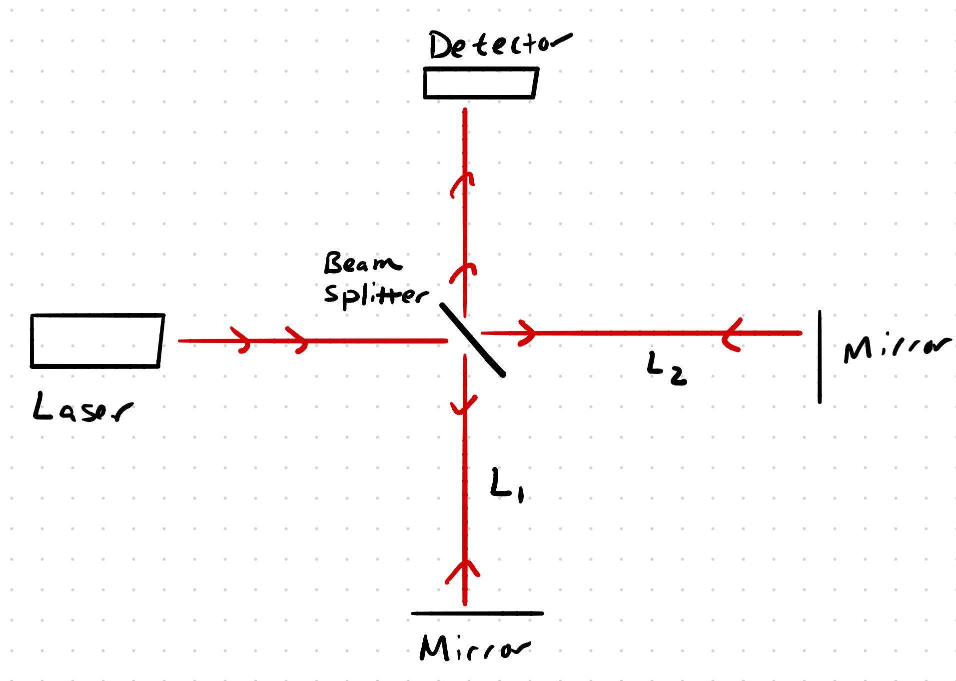

Figure 5: Interferometer optical setup

Because of these observations, we can use Eqn 10 to compute the intensity at a point in the screen. All that has to be done is to compute the difference in distance to the slits, $\delta = r_1 - r_2$.

The calculation of δ can be done by assuming that L is much larger than the separation between the slits, d. In that case, $\delta = d \sin\theta$. So the angles at which the intensity is maximal will be when $d \sin\theta = m\lambda$, with m an integer. There will be minima between each of the maxima!

Example: Interferometer

The interferometer uses interference to measure distances very accurately, as you might be able to tell from the name (interfero - meter). In this setup, a laser is sent down a path where it encounters a half-silvered mirror set at a 45 degree angle. This mirror reflects half of the light and allows half of the light to pass through. The **intensity** is reduced by a factor of two for each path, but then recombines. The light then travels down two separate paths, of length L_1 and L_2, and rejoin at the mirror. We place a detector as noted in the figure.

Since the mirrors typically absorb some of the light, we typically don't work the intensity of the laser, but instead use the intensity of the light with one path blocked: I_1. Since the two paths are in phase, we can use Eqn 10, so

$$I = 4I_1 \cos^2\left(k\frac{r_1 - r_2}{2}\right). \tag{11}$$

r_1 is the distance traveled on one path, and r_2 is the distance traveled on the second path. It actually doesn't matter the total values of r_1 and r_2; we can compute $r_1 - r_2$ knowing just the *difference* between the arm lengths:

$$r_1 - r_2 = 2(L_1 - L_2) \tag{12}$$

The factor of two occurs because the light travels the arms twice, once to the terminal mirror, and again on the way back.