Unit 5: Probability and complex numbers

At this point, we note that we have a tension: we describe light as a wave in that it exhibits interference, but light arrives in what appears to be discrete packets that come at random. This is sometimes referred to as "wave-particle duality;" however, the real answer is much more revolutionary. The resolution to this duality is that everything is described using *probability waves* that interfere just like the waves we have been studying earlier in the class, and allow us to compute the probabilities of events (such as observing a photon at a particular location).

The probability waves are written in terms of complex numbers, and are used to compute probabilities. At this point in their career, many students have not had a lot of experience with this mathematics, so we will spend this unit discussing these concepts.

After this unit, you should be able to

- Given a probability density $\rho(x)$ for the position of a particle, compute the probability of observing that particle within a given range $a < x < b$.
- Using the probability for a particle of a given kinetic energy hitting a detector and the flux of particles, compute the total power incident on the detector.
- Manipulate complex numbers to find the magnitude squared and complex conjugate, and use Euler's equation.

Probability density

A probability is a number between 0 and 1. A probability density is a function, often called $\rho(x)$, that represents the probability **per unit length**. This is similar to the relationship between intensity and power; intensity is the power per unit area, and the power is the total amount of energy per second.

Probability densities have the following properties:

$$\rho(x) \geq 0 \tag{1}$$

and so-called normalization

$$\int_{-\infty}^{\infty} \rho(x)dx = 1. \tag{2}$$

Normalization ensures that the probability of the particle being *somewhere* is equal to 1.

The probability for x to be between two points a and b, assuming $a < b$, is

$$P(a < x < b) = \int_a^b \rho(x)dx. \tag{3}$$

Because of Eqns. 1 and 2, this probability is always between 0 and 1. Note that ρ can actually have a value greater than 1, as long as it is normalized.

Probability density examples

Normalization

Suppose a probability density is given by $\rho(x) = Ne^{-x}$ for $0 < x < \infty$, and is zero elsewhere. What must N be to ensure the probability density is normalized?

Solution: We must have

$$\int_{-\infty}^{\infty} \rho(x) dx = \int_{0}^{\infty} Ne^{-x} dx = 1. \tag{4}$$

The integral starts at zero because $\rho = 0$ for $x < 0$. The integral

$$\int_{0}^{\infty} e^{-x} dx = -e^{-x}|_0^\infty = 1. \tag{5}$$

So therefore $N \cdot 1 = 1$ and $N = 1$ to normalize this probability density.

Number of particles per second

Suppose that we place a detector between $x = a$ and $x = b$ μm. Suppose that the normalized probability density of a particle hitting the detector in that region is given by $\rho(x) = (0.1 + Cx)$ μm^{-1} in that region, with $C = 0.05$ μm^{-2}. The probability density must have units of inverse length because when we integrate it, it must equal a unitless number.

Question part 1

What is the probability that a single particle hits the detector?

Solution: The probability is given by

$$P(a < x < b) = \int_a^b 0.1 + Cx dx = 0.1(b - a) + \frac{C}{2}(b^2 - a^2) \tag{6}$$

Question part 2

Suppose that 1000 particles are sent at the detector per second. How many will hit the detector on average per second?

Solution: The number is $1000 \cdot P(a < x < b)$ particles per second, since each particle has probability $P(a < x < b)$ to hit the detector.

Complex numbers

In quantum mechanics, we describe the interference of particles using complex numbers. This is very similar to the phasor description of waves. Some rules:

- $i = \sqrt{-1}$.

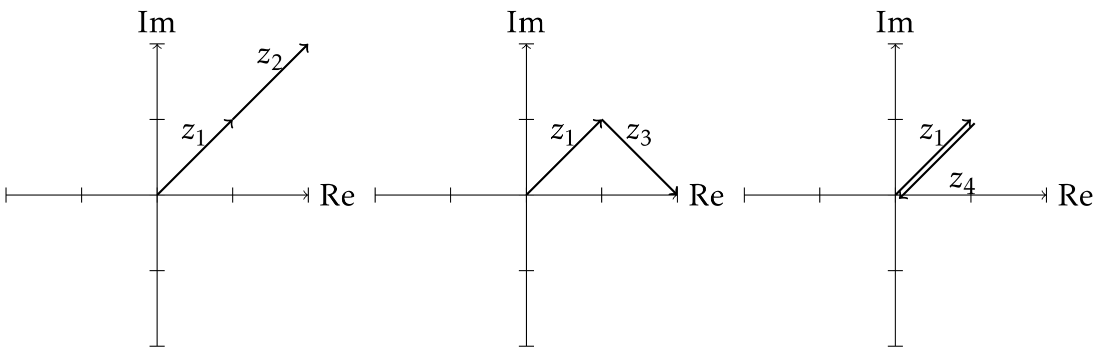

Figure 1: Adding complex numbers. For $z = x + iy$, the real part is x, labeled Re, and the imaginary part is y, labeled Im.

- $e^{i\theta} = \cos(\theta) + i\sin(\theta)$
- For a complex number $z = x + iy$, the complex conjugate $z^* = x - iy$
- For a complex number $z = x+iy$, the magnitude squared (also known as the absolute value squared) is $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 + y^2$

For complex conjugation, the main thing is to remember that the i gets a minus sign.

We will sometimes write complex numbers as $Ae^{i\theta}$, where A is some positive real overall amplitude. This has the advantage that the magnitude squared is $Ae^{i\theta}Ae^{-i\theta} = A^2$.

You can think of complex numbers as being better phasors. As implied by $z = x + iy$, the numbers can be drawn on a 2D plot, and added component-wise in the same way as 2D vectors:

$$z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2) \tag{7}$$

The main difference between complex numbers and phasors is that we can multiply complex numbers to get another complex number:

$$z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 - y_1y_2 \tag{8}$$

Example: interference using complex numbers

Consider the four complex numbers:

$$\begin{aligned} z_1 &= 1 + i \\ z_2 &= 1 + i \\ z_3 &= 1 - i \\ z_4 &= -1 - i \end{aligned}$$

Some sums are shown in Fig 1. What is the magnitude squared of the complex numbers $z_{1i} = z_1 + z_i$ for $i = 2, 3, 4$?

Solution: First let's compute the sums

$$z_{12} = 2 + 2i,$$
$$z_{13} = 2,$$
$$z_{14} = 0.$$

Now using the definition of the magnitude squared,

$$|z_{12}|^2 = z_{12}z_{12}^* = (2+2i)*(2-2i) = 4+4 = 8, \tag{9}$$
$$|z_{13}|^2 = z_{13}z_{13}^* = (2)*(2) = 4, \tag{10}$$
$$|z_{14}|^2 = z_{14}z_{14}^* = (0)*(0) = 0. \tag{11}$$

You may be able to see the reason for the definition $|z|^2 = zz^*$; it is equivalent to the definition of length for a 2D vector $|v|^2 = v_x^2 + v_y^2$.

Example 2: Euler's equation

We can also write $z = Re^{i\theta} = R\cos\theta + iR\sin\theta$, with R a positive real number. This is the equivalent of writing a 2D vector in polar coordinates. Show that $|z|^2 = R^2$.

Solution: We must use the identity $z^* = Re^{-i\theta}$, where we used the rule of replacing i with $-i$ to take the complex conjugate. Then

$$|z|^2 = zz^* = Re^{i\theta}Re^{-i\theta}. \tag{12}$$

Using the identity $e^a e^b = e^{a+b}$,

$$|z|^2 = R \cdot R = R^2, \tag{13}$$

which is the desired relationship.