Physics 508
Handout 11 Mathematical Methods in Physics I
Fall 2022 Homework 11 University of Illinois

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1) Dielectric Sphere: Consider a solid dielectric sphere of radius $a$ and permittivity $\varepsilon$. The sphere is placed in an electric field which takes the constant value $E_0\hat{z}$ a long distance from the sphere. Recall that Maxwell’s equations require that $D_\perp$ and $E_\parallel$ be continuous across the surface of the sphere.

a) Use the expansions

$$\Phi_{in} = \sum_l A_l r^l P_l(\cos \theta)$$
$$\Phi_{out} = \sum_l (B_l r^l + C_l r^{-l-1}) P_l(\cos \theta)$$

and find all non-zero coefficients $A_l$, $B_l$, $C_l$.

b) Show that the $E$ field inside the sphere is uniform and of magnitude $3\frac{\varepsilon_0}{\varepsilon + 2\varepsilon_0}E_0$.

c) Show that the electric field is unchanged if the dielectric is replaced by the polarization-induced surface charge density

$$\sigma_{induced} = 3\varepsilon_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) E_0 \cos \theta.$$  

(Some systems of units may require extra $4\pi$’s in this last expression. In SI units $D \equiv \varepsilon E = \varepsilon_0 E + P$, and the polarization induced charge density is $\rho_{induced} = -\nabla \cdot P$)

2) Hollow Sphere: The potential on a spherical surface of radius $a$ is $\Phi(\theta, \phi)$. We want to express the potential inside the sphere as an integral over the surface in a manner analogous to the Poisson kernel in two dimensions.

a) By using the generating function for Legendre polynomials, show that

$$\frac{1 - r^2}{(1 + r^2 - 2r \cos \theta)^{3/2}} = \sum_{l=0}^\infty (2l + 1)r^l P_l(\cos \theta), \quad r < 1$$

b) Starting from the expansion

$$\Phi_{in}(r, \theta, \phi) = \sum_{l=0}^\infty \sum_{m=-l}^l A_{lm} r^l Y^l_m(\theta, \phi)$$

$$A_{lm} = \frac{1}{a^l} \int_{S^2} \left[Y^l_m(\theta, \phi)\right]^* \Phi(\theta, \phi) \cos \theta \cos \phi$$

and using the addition formula for spherical harmonics, show that

$$\Phi_{in}(r, \theta, \phi) = \frac{a(a^2 - r^2)}{4\pi} \int_{S^2} \frac{\Phi(\theta', \phi')}{(r^2 + a^2 - 2ar \cos \gamma)^{3/2}} d\cos \theta' d\phi'$$

where $\cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. 

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c) By setting \( r = 0 \), deduce that a three dimensional harmonic function cannot have a
local maximum or minimum.

3) Peierls Problem: Computing the Critical Mass. The core of a fast breeder reactor
consists of a sphere of fissile \(^{235}\text{U} \) of radius \( R \). It is surrounded by a thick shell of non-fissile
material which acts as a neutron reflector, or \textit{tamper}.

\[
\text{Fast breeder reactor.}
\]

In the core, the fast neutron density \( n(r, t) \) obeys

\[
\frac{\partial n}{\partial t} = \nu n + D_F \nabla^2 n.
\]

Here the term with \( \nu \) accounts for the production of additional neutrons due to induced
fission. The term with \( D_F \) describes the diffusion of the fast neutrons. In the tamper the
neutron flux obeys

\[
\frac{\partial n}{\partial t} = D_T \nabla^2 n.
\]

Both the neutron density, \( n \), and the neutron flux, \( j = D_{F,T} \nabla n \), are continuous across the
interface between the two materials. Find an equation determining the critical radius, \( R_c \),
above which the neutron density grows without bound. Show that the critical radius for
an assembly with a tamper consisting of \(^{238}\text{U} \) \( (D_T = D_F) \) is one-half of that for a core
surrounded only by air \( (D_T = \infty) \), and so the use of a thick \(^{238}\text{U} \) tamper reduces the critical
mass by a factor of eight.

4) Bessel functions and impact parameters: In two dimensions we can expand a plane
wave as

\[
e^{iky} = \sum_{n=-\infty}^{\infty} J_n(kr)e^{in\theta}.
\]

a) What do you think the resultant wave will look like if we take only a finite segment of
this sum? For example

\[
\phi(x) = \sum_{l=10}^{17} J_n(kr)e^{in\theta}.
\]
Think about:

i) The quantum interpretation of $\hbar l$ as angular momentum $= \hbar kd$, where $d$ is the *impact parameter*, the amount by which the incoming particle misses the origin.

ii) Diffraction: one cannot have a plane wave of finite width.

b) After (I trust you!) writing down your best guess for the previous part, confirm your understanding by using Mathematica or other package to plot the real part of $\phi$ as defined above. The following Mathematica code may work.

```mathematica
Clear[bit,tot]
bit[l_,x_,y_]:=Cos[l ArcTan[x,y]]BesselJ[l,Sqrt[x^2+y^2]]
tot[x_,y_]:=Sum[bit[l,x,y],{l,10,17}]
ContourPlot[tot[x,y],{x,-40,40},{y,-40,40},PlotPoints ->200]
Display["wave",%,"EPS"]
```

Try different ranges for the sum. Include a printout of your code together with the plot.

5) Berry’s problem: Levitating Frogs (Optional). Sir Michael Berry (of Berry’s phase fame) shared the 2000 IgNobel Prize for his work establishing the theoretical basis for the diamagnetic levitation of frogs. The picture shows a live frog floating happily in the bore of a 16 T Bitter Solenoid at the High Field Magnet Lab in the University of Nijmegen. The experiment was performed by Andre Geim, who shared the IgNobel with with Berry and the won the real Nobel prize in 2010 for his work on graphene — another strongly diamagnetic material. (A movie of the floating frog is at https://www.youtube.com/watch?v=A1vyB-O5i6E).

![Levitated frog.](image)

a) Earnshaw’s theorem: Show that an object that experiences the combined force of gravity and a magnetic force $\propto B$, cannot have a position of stable equilibrium.

b) Show that, in a current-free region, the magnitude of the magnetic field obeys

$$\nabla^2 |B|^2 = 2(|\nabla B_x|^2 + |\nabla B_y|^2 + |\nabla B_z|^2) \geq 0.$$

c) Deduce that $|B|^2$ can never have a local maximum, but may have a local minimum.
d) Consider the field in the bore of a cylindrical solenoid. Write $\mathbf{B} = \nabla \phi$, where $\phi$ is the magnetic potential, and define $\phi_n(z) = \partial^n_z \phi(0, 0, z)$, the derivatives along the bore axis. Use cylindrical symmetry, together with the fact that $\phi$ is harmonic, to show that near the bore axis

$$\phi(x, y, z) \approx \phi_0(z) - \frac{1}{4}(x^2 + y^2)\phi_2(z),$$

and hence

$$|\mathbf{B}|^2 \approx \phi_1^2(z) + \frac{1}{4}(x^2 + y^2)(\phi_2^2(z) - 2\phi_1(z)\phi_3(z)).$$

Deduce (Berry and Gaim) that the second derivatives of $|\mathbf{B}|^2$ are all positive when

$$D_1(z) \equiv B'(z)^2 + B(z)B''(z) > 0, \quad \text{vertical stability},$$
$$D_2(z) \equiv B'(z)^2 - 2B(z)B''(z) > 0, \quad \text{horizontal stability}.$$

Here $B(z) = |\mathbf{B}(0, 0, z)|$ is the magnitude of the field on the bore axis. Observe that both these inequalities will be satisfied in some vicinity of a point of inflection of $B(z)$.

e) Deduce that a \emph{paramagnetic} object, which is attracted to regions of strong $|\mathbf{B}|^2$, can never be stably levitated, but that a \emph{diamagnetic} frog — which is predominantly water — may be stably levitated.