1) **Missing State:** In Homework Set 4 you found that the Schrödinger equation
\[
\left( -\frac{d^2}{dx^2} - 2 \text{sech}^2 x \right) \psi = E \psi
\]
has eigensolutions
\[
\psi_k(x) = e^{ikx}(-ik + \tanh x)
\]
with eigenvalue \(E = k^2\).

- Show that for \(x\) large and positive \(\psi_k(x) \approx A e^{ikx} e^{i\delta(k)}\), while for \(x\) large and negative \(\psi_k(x) \approx A e^{ikx} e^{-i\delta(k)}\), the complex constant \(A\) (which you must write down explicitly) being the same in both cases. Express \(\delta(k)\) as the inverse tangent of an algebraic expression in \(k\).
- Impose periodic boundary conditions \(\psi(-L/2) = \psi(+L/2)\) where \(L \gg 1\). Find the allowed values of \(k\) and hence an explicit expression for the \(k\)-space density, \(\rho(k) = \frac{dn}{dk}\), of the eigenstates.
- Compare your formula for \(\rho(k)\) with the corresponding expression, \(\rho_0(k) = L/2\pi\), for the eigenstate density of the zero-potential equation and compute the integral
\[
\Delta N = \int_{-\infty}^{\infty} \{\rho(k) - \rho_0(k)\} dk.
\]
- Deduce that one eigenfunction has gone missing from the continuum and presumably become a localized bound state. (You will have found an explicit expression for this localized eigenstate in Homework Set 4.)

2) **Continuum Completeness:** Consider the differential operator
\[
\hat{L} = -\frac{d^2}{dx^2}, \quad 0 \leq x < \infty
\]
with self-adjoint boundary conditions \(\psi(0)/\psi'(0) = \tan \theta\) for some fixed angle \(\theta\).

- Show that when \(\tan \theta < 0\) there is a single normalizable negative-eigenvalue eigenfunction localized near the origin, but none when \(\tan \theta > 0\).
- Show that there is a continuum of positive-eigenvalue eigenfunctions of the form \(\psi_k(x) = \sin(kx + \delta(k))\) where the phase shift \(\delta\) is found from
\[
e^{i\delta(k)} = \frac{1 + ik \tan \theta}{\sqrt{1 + k^2 \tan^2 \theta}}.
\]
- Write down (no justification required) the appropriate completeness relation
\[
\delta(x - x') = \int\frac{dn}{dk} N_k \psi_k(x) \psi_k(x') \, dk + \sum_{\text{bound}} \psi_n(x) \psi_n(x')
\]
with an explicit expression for the product (not the separate factors) of the density of states and the normalization constant $N_k$, and with the correct limits on the integral over $k$.

- Confirm that the $\psi_k$ continuum on its own, or together with the bound state when it exists, form a complete set. You will do this by evaluating the integral

$$I(x, x') = \frac{2}{\pi} \int_0^\infty \sin(kx + \delta(k)) \sin(kx' + \delta(k)) \, dk$$

and interpreting the result. You will need the following standard integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{1}{1 + k^2t^2} = \frac{1}{2|t|} e^{-|x|/|t|}.$$

To get full credit, you must show how the bound state contribution switches on and off with $\theta$. The modulus signs are essential for this.

3) Fredholm Alternative:

A heavy elastic bar with uniform mass $m$ per unit length lies almost horizontally. It is supported by a distribution of upward forces $F(x)$.

The shape of the bar, $y(x)$, can be found by minimizing the energy

$$U[y] = \int_0^L \left\{ \frac{1}{2} \kappa (y'')^2 - (F(x) - mg) y \right\} dx,$$

which gives (homework 2!) the equation

$$\hat{L} y \equiv \kappa \frac{d^4 y}{dx^4} = F(x) - mg, \quad y'' = y''' = 0 \quad \text{at} \quad x = 0, L.$$

- Show that the boundary conditions are such that the operator $\hat{L}$ is self-adjoint with respect to an inner product with weight function 1.
- Find the zero modes which span the null space of $\hat{L}$.
- If there are $n$ linearly independent zero modes, then the codimension of the range of $\hat{L}$ is also $n$. Using your explicit solutions from the previous part, find the conditions that must be obeyed by $F(x)$ for a solution of $\hat{L} y = F - mg$ to exist. What is the physical meaning of these conditions?
- The solution to the equation and boundary conditions is not unique. Is this non-uniqueness physically reasonable? Explain.