

Solutions to Homework Set 2

1) **Snellius' law:** Parts a), b) are trivial, so I will not write out the solutions. Part c) has two sub-parts:

i) Setting the variation of $F_1[y]$ to zero gives

$$\frac{d}{dx} \left(n(x) \frac{y'}{\sqrt{1+y'^2}} \right) = 0.$$

Now elementary calculus tells us that $y' = \tan \theta$, where θ is the angle between the light ray and the x -axis. The x -axis is, however, for this geometry, the normal to the planes of constant $n(x)$. Thus we have $\theta = \psi$, where ψ is the angle of incidence as it is usually defined in optics. Using the standard trig identities $\sec^2 \theta - \tan^2 \theta = 1$, and $\sec \theta = 1/\cos \theta$, we reduce our equation to

$$\frac{d}{dx} \left(n(x) \sin \psi \right) = 0,$$

or

$$n(x) \sin \psi = \text{const.}$$

This last equation is the usual form of Snell's law.

ii) For $F_2[y]$ we can use the first integral to deduce directly that

$$n(y) \frac{y'^2}{\sqrt{1+y'^2}} - n(y) \sqrt{1+y'^2} = \text{const.},$$

or, collecting terms,

$$\frac{n(y)}{\sqrt{1+y'^2}} = \text{const.}$$

Again using $y' = \tan \theta$ and $\sec^2 \theta - \tan^2 \theta = 1$, this reduces to

$$n(y) \cos \theta = \text{const.}$$

In this geometry the angle of incidence is the complement of θ , so $\cos \theta = \sin \psi$, and again Snell's law is obtained.

2) **Lobachevski Geometry:** The first integral arising from the functional $F_3[y]$ is

$$\frac{1}{y} \frac{y'^2}{\sqrt{1+y'^2}} - \frac{1}{y} \sqrt{1+y'^2} = \text{const.}$$

We group the terms and invert to get

$$y \sqrt{1+y'^2} = C,$$

whence

$$y^2 \left(\frac{dy}{dx} \right)^2 = C^2 - y^2.$$

Separating variables, we have

$$\frac{ydy}{\sqrt{C^2 - y^2}} = dx.$$

Performing the integral, we find

$$-\sqrt{C^2 - y^2} = x + a,$$

or equivalently

$$(x + a)^2 + y^2 = C^2.$$

This is manifestly the equation of a circle radius C with its centre on the x axis at $x = -a$.

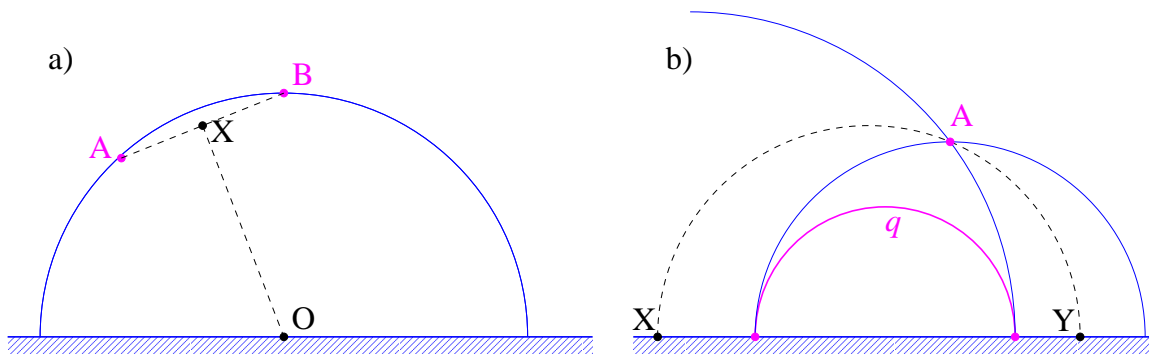


Figure a) shows the construction of the unique line through the points A and B: at the midpoint X of AB construct the perpendicular bisector XO of AB such that O lies on the x axis. Then a circle centered at O and passing through A necessarily passes through B.

Figure b) shows the line q and, in blue, the two lines passing through the given point A and parallel to q . The line XY is one of infinitely many lines passing through A and not intersecting q . All lines outside the region bounded by the two parallel lines will intersect q .

3) Drums and Membranes: The first step is to derive an expression for the surface area of the drumhead. This will probably be familiar to you from multivariate calculus. Consider an infinitesimal square in the x - y plane with sides dx , dy . The function $h(x, y)$ maps this to a parallelogram defined by the pair of infinitesimal vectors $(dx, 0, \partial h/\partial x dx)$ and $(0, dy, \partial h/\partial y dy)$. The magnitude of the vector product of these vectors is

$$\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2} dx dy.$$

This is the area of the drum surface lying above the infinitesimal square. The total area is therefore

$$\text{Area} = \int dx dy \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}.$$

In the limit that the partial derivatives of h are small, thus reduces to $\int dx dy |\nabla h|^2/2$.

We consider the variation

$$\begin{aligned} \delta \left(\int dx dy \frac{1}{2} |\nabla h|^2 \right) &= \int dx dy (\nabla h \cdot \nabla \delta h) \\ &= \int dx dy (\nabla \cdot (\delta h \nabla h) - \delta h \nabla^2 h) \\ &= - \int dx dy (\delta h \nabla^2 h), \end{aligned}$$

where I've ignored the boundary contribution coming from the integral of the divergence because a drum has its rim fixed. The area will therefore be stationary provided that $\nabla^2 h = 0$.

The potential energy of the drumskin is $T \int dx dy |\nabla h|^2 / 2$, and its kinetic energy is $\rho \int dx dy \dot{h}^2 / 2$. The lagrangian for the drumhead is therefore

$$\int dx dy \left\{ \frac{1}{2} \rho \left(\frac{\partial h}{\partial t} \right)^2 - \frac{T}{2} |\nabla h|^2 \right\}.$$

To find the equation of motions we compute

$$\begin{aligned} \delta S &= \int dt dx dy \left\{ \rho \frac{\partial h}{\partial t} \frac{\partial \delta h}{\partial t} - T \frac{\partial h}{\partial x} \frac{\partial \delta h}{\partial y} - T \frac{\partial h}{\partial y} \frac{\partial \delta h}{\partial x} \right\} \\ &= \int \int dt dx dy \delta h(t, x, y) \left\{ \rho \frac{\partial^2 h}{\partial t^2} - T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) \right\} \end{aligned}$$

where, in passing from the first line to the second, we have discarded the total divergence

$$\frac{\partial}{\partial t} \left(\rho \delta h \frac{\partial h}{\partial t} \right) + \frac{\partial}{\partial x} \left(-T \delta h \frac{\partial h}{\partial y} \right) + \frac{\partial}{\partial y} \left(-T \delta h \frac{\partial h}{\partial x} \right).$$

Thus $\delta S = 0$ implies the two-dimensional wave equation

$$\rho \frac{\partial^2 h}{\partial t^2} - T \left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} \right) = 0.$$

4) Magnetostatics:

- i) The first term in the integrand is clearly gauge invariant because the curl of a gradient is zero. The change in the second term is

$$\int_{\mathbb{R}^3} -\mathbf{J} \cdot \nabla \chi d^3 x.$$

Now we use the vector identity

$$\nabla \cdot (\chi \mathbf{J}) = (\nabla \chi) \cdot \mathbf{J} + \chi (\nabla \cdot \mathbf{J})$$

to write this as

$$\int_{\mathbb{R}^3} \chi \nabla \cdot \mathbf{J} d^3 x = 0.$$

We have discarded the integral of the divergence, because we know that $\mathbf{J} = 0$ on the boundary at infinity.

ii) We have

$$\delta F = \int_{\mathbb{R}^3} \left\{ \frac{1}{\mu(x)} (\nabla \times \delta \mathbf{A}) \cdot (\nabla \times \mathbf{A}) - \mathbf{J} \cdot \delta \mathbf{A} \right\} d^3x$$

Now we use the vector identity

$$\nabla \cdot (\delta \mathbf{A} \times \mathbf{H}) = (\nabla \times \delta \mathbf{A}) \cdot \mathbf{H} - \delta \mathbf{A} \cdot (\nabla \times \mathbf{H})$$

with $\mathbf{H} = \mu^{-1} \nabla \times \mathbf{A}$. We may discard the integral of the divergence because $|\mathbf{H}| = O(R^{-3})$ at large distance, and so have

$$\delta F = \int_{\mathbb{R}^3} \left\{ \delta \mathbf{A} \cdot \left(\nabla \times \left(\frac{1}{\mu(x)} (\nabla \times \mathbf{A}) \right) - \mathbf{J} \right) \right\} d^3x$$

Thus $\delta F = 0$ gives the Maxwell equation $\nabla \times \mathbf{H} = \mathbf{J}$, *i.e.* Ampère's law.

iii) We write $\mathbf{J} = \nabla \times \mathbf{H}_{\text{physical}}$ in F , so that it becomes

$$F[\mathbf{A}] = \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu} |\mathbf{B}|^2 - \mathbf{A} \cdot (\nabla \times \mathbf{H}_{\text{physical}}) \right\} d^3x$$

Now

$$\nabla \cdot (\mathbf{H}_{\text{physical}} \times \mathbf{A}) = \mathbf{A} \cdot (\nabla \times \mathbf{H}_{\text{physical}}) - \mathbf{H}_{\text{physical}} \cdot (\nabla \times \mathbf{A})$$

so, with $\mathbf{B} = \mu \mathbf{H}$ as always, we have

$$\begin{aligned} F[\mathbf{A}] &= \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu} |\mathbf{B}|^2 - \frac{1}{\mu} \mathbf{B}_{\text{physical}} \cdot \mathbf{B} \right\} d^3x \\ &= \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu} |\mathbf{B}|^2 - \frac{1}{\mu} \mathbf{B}_{\text{physical}} \cdot \mathbf{B} \right\} d^3x \\ &= \int_{\mathbb{R}^3} \left\{ \frac{1}{2\mu} |\mathbf{B} - \mathbf{B}_{\text{physical}}|^2 - \frac{1}{2\mu} |\mathbf{B}_{\text{physical}}|^2 \right\} d^3x \end{aligned}$$

and (provided that $\mu(x)$ is everywhere positive) this clearly takes its minimum value

$$F_{\text{minimum}} = - \int_{\mathbb{R}^3} \frac{1}{2\mu} |\mathbf{B}_{\text{physical}}|^2 d^3x$$

whenever $\mathbf{B} \equiv \nabla \times \mathbf{A} = \mathbf{B}_{\text{physical}}$.