Solutions to Homework Set 4

Test functions and distributions: For part a) we take any test function $\varphi(x)$ and look at

$$\begin{aligned}(\varphi, f\delta' + f'\delta) &\equiv \int_{-\infty}^{\infty} \varphi(x)\left\{f(x)\delta'(x) + f'(x)\delta(x)\right\} dx \\ &= \int_{-\infty}^{\infty} \left\{[-\varphi'(x)f(x) - \varphi(x)f'(x)]\delta(x) + \varphi(x)f'(x)\delta(x)\right\} dx \\ &= -\varphi'(0)f(0),\end{aligned}$$

and compare it with

$$\begin{aligned}(\varphi, f(0)\delta') &\equiv \int_{-\infty}^{\infty} \varphi(x)\left\{f(0)\delta'(x)\right\} dx \\ &= -\varphi'(0)f(0).\end{aligned}$$

The results are the same, and so the distributions $f\delta' + f'\delta$ and $f(0)\delta'$ are equal.

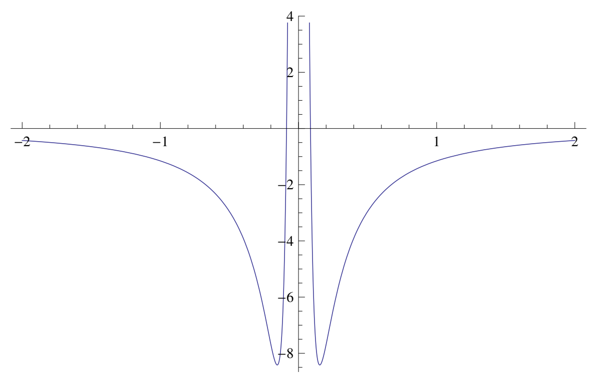

Figure 1: A plot of $\delta_\mu^{(1/2)}(x)$ for $\mu = .05$.

For b) we note that for $x \gg \mu$ the function $\delta_\mu^{(1/2)}(x)$ behaves as $-|x|^{-3/2}/\sqrt{8\pi}$. The function is zero at $|x| = \sqrt{3}\mu$, and for $|x|$ less than this we have a large postive spike. This spike will give a contribution $\varphi(0) \times$ Area to the integral. Here "Area" is the area under the spike. This area grows as $\mu \to 0_+$, so it is not tending to a delta function. We know, however, that the total area under the graph of $\delta_\mu^{(1/2)}(x)$ is *zero*. This is because the Fourier transform is zero at $k = 0$. Our integral is therefore

$$\int_{|x|>\sqrt{3}\mu} \delta_\mu^{(1/2)}(x)(\varphi(x) - \varphi(0))\, dx.$$

With the aid of the subtraction, the integral is convergent (although improper) as $\mu \to 0_+$. If you want to be rigorous, you can say that away from the spike,

$$\delta_\mu^{(1/2)}(x) \to -\frac{1}{\sqrt{8\pi}}|x|^{-3/2},$$

with pointwise convergence from above. We can therefore appeal to Lebesgue's dominated convergence theorem to complete the proof.

For (c) we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left\{P\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x-t)}\,dx\right\} &= \frac{\partial}{\partial t}\left\{\left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{\varphi(x)}{(x-t)}\,dx\right\}\\
&= \frac{\varphi(t-\epsilon)}{-\epsilon}-\frac{\varphi(t+\epsilon)}{\epsilon}+\left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{\varphi(x)}{(x-t)^2}\,dx\\
&= -\frac{2\varphi(t)}{\epsilon}+\left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{\varphi(x)}{(x-t)^2}\,dx+O(\epsilon)\\
&= \left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,dx+O(\epsilon)\\
&\to P\int_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,dx \quad \text{as } \epsilon\to 0.
\end{aligned}
$$

The tricky part was recognizing that

$$\varphi(t+\epsilon)+\varphi(t-\epsilon)=2\varphi(t)+O(\epsilon^2),$$

(*i.e.* that the error is of order ϵ^2 and not of order ϵ) and that you can rewrite the resultant divergent bit $2\varphi(t)/\epsilon$ as

$$\frac{2\varphi(t)}{\epsilon}=\varphi(t)\left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{1}{(x-t)^2}\,dx=\left(\int_{-\infty}^{t-\epsilon}+\int_{t+\epsilon}^{\infty}\right)\frac{\varphi(t)}{(x-t)^2}\,dx.$$

Note that in making the estimate

$$\varphi(t+\epsilon)+\varphi(t-\epsilon)=2\varphi(t)+O(\epsilon^2)$$

were are not claiming that φ has a convergent Taylor expansion, but are instead appealing to the Taylor-series with remainder

$$\varphi(t+\epsilon)=\varphi(t)+\epsilon\varphi'(t)+\frac{1}{2}\epsilon^2\varphi''(\xi), \quad \xi\in[t,t+\epsilon],$$

which only requires that φ possess a second derivative everywhere in $[t,t+\epsilon]$. Because φ is smooth this second derivative is itself continuous, and hence bounded, in the interval.

2) One-dimensional scattering theory: This problem exploits the fact that the Wronskian of any pair of solutions of a Schrödinger equation is independent of x. The four solutions having the same energy, and therefore being solutions of the same equation, are $\psi_{\pm k}$ and $\psi^*_{\pm k}$. The Wronskians of any pair of these are the same when we evaluate them to the left ($x\in L$) or to right ($x\in R$) of the scatterer.

a) We have, taking $k>0$,

$$W(\psi_k^*,\psi_k)=\begin{cases}2ik(1-|r_L|^2) & x\in L,\\ 2ik|t_L|^2 & x\in R.\end{cases}$$

Thus $1-|r_L|^2=|t_L|^2$. This method applied to states with $k<0$ leads to the same result, but with $L\to R$.

b) Let $k > 0$ and consider

$$W(\psi_{-k}, \psi_k) = \begin{cases} 2ikt_R(-k) & x \in L, \\ 2ikt_L(k) & x \in R. \end{cases}$$

Thus $t_R(-k) = t_L(k)$.

d) The equations to be satisfied at the delta-function are

$$\begin{aligned} \psi(a-\epsilon) &= \psi(a+\epsilon), \quad \text{continuity,} \\ \psi'(a+\epsilon) - \psi'(a-\epsilon) &= \lambda\psi(a), \qquad \text{slope jump.} \end{aligned}$$

Plugging in the given expressions gives, assuming $k > 0$,

$$\begin{aligned} e^{ika} + r_L(k)e^{-ika} &= t_L(k)e^{ika}, \\ ikt_L(k)e^{ika} - ik\left(e^{ika} - r_L(k)e^{-ika}\right) &= \lambda t_L(k)e^{ika}. \end{aligned}$$

Solving for the reflection and transmision coefficients, leads to

$$r_L(k) = e^{2ika}\frac{\lambda}{2ik-\lambda}, \quad t_L(k) = \frac{2ik}{2ik-\lambda}.$$

Similarly, taking $k < 0$, we find

$$r_R(-k) = e^{-2ika}\frac{\lambda}{2ik-\lambda}, \quad t_R(-k) = \frac{2ik}{2ik-\lambda}.$$

Thus, if $a \neq 0$, $r_L(k)$ and $r_R(-k)$ differ by a phase.

3) Reduction of Order:

a) We have been given a solution $y_1(x) = u$ and seek a second solution in the form $y_2 = uv$. Plug $y = uv$ into $y'' + Vy = 0$ to get

$$v(u'' + Vu) + 2u'v' + v''u = 0.$$

Thus uv is a solution of the equation provided $2u'v' + uv'' = 0$, or

$$2\frac{u'}{u} + \frac{v''}{v'} = 0.$$

This is equivalent to

$$\frac{d}{dx}\left(\ln u^2 + \ln v'\right) = 0,$$

which is in turn equivalent to

$$v' = \frac{const.}{u^2}.$$

We may as well take the constant to be unity, and then

$$v(x) = \int_a^x \frac{d\xi}{u^2(\xi)}.$$

The lower limit is irrelevent, because changing it merely adds a constant to v and thus a multiple of y_1 to y_2. We have therefore established that

$$y_2 = u(x) \int^x \frac{d\xi}{u^2(\xi)}$$

is a solution of $y'' + V(x)y = 0$.

The wronskian of the old and new solutions is

$$W(y_1, y_2) = u(x)\left(u'(x)\int^x \frac{d\xi}{u^2(\xi)} + \frac{1}{u(x)}\right) - \left(u(x)\int^x \frac{d\xi}{u^2(\xi)}\right)u'(x) = 1.$$

This is not zero, so the two solutions are linearly independent.

b) This problem looks difficult at first sight, but is in fact one of those labyrinths in which at each turning there is only one direction in which to proceed, and so no clew is required.

We begin by observing that we can, without loss of generality, take $y_1 y_2 = 1$. Now the only possible action we can take at this stage is to differentiate this expression. We get $y_1' y_2 + y_1 y_2' = 0$. This does not seem help much, so the only possible thing to do is to differentiate again. Now we get

$$y_1'' y_2 + 2y_1' y_2' + y_1 y_2'' = 0 \quad \Rightarrow \quad -2p_2 y_1 y_2 + 2y_1' y_2' = 0.$$

Here we have been able to use the differential equation obeyed by y_1 and y_2, together with $y_1' y_2 + y_1 y_2' = 0$, to simplify the result. We now use $y_1 y_2 = 1$ to deduce that $y_1' y_2' = p_2$. Progress!! Differentiate yet once more and use the same tricks and we are are home:

$$p_2' = -2p_1 p_2.$$

In passing we have also discovered what y_1, y_2 must be. We found

$$p_2 = y_1' y_2' = -\left(\frac{y_1'}{y_1}\right)^2 = -[(\ln y_1)']^2,$$

where we have used that $y_1 y_2 = 1 \Rightarrow \ln y_1 = -\ln y_2$. We therefore take the square root and integrate to find

$$y_1(x) = \exp\left\{\pm \int_a^x \sqrt{-p_2(\xi)}\, d\xi\right\}.$$

Chose one of the $\pm$ signs, and then y_2 is the same expression with the other choice of sign.

c) The equation

$$(x+1)x^2 y'' + xy' - (x+1)^3 y = 0$$

satisfies the relation $p_2' = -2p_1 p_2$, and has $p_2 = -(1+x)^2/x^2$. Thus

$$y(x) = \exp\left\{\pm \int_a^x (1 + \xi^{-1})\, d\xi\right\} \propto x^{\pm 1} e^{\pm x}.$$

4) Schwarzian Derivative:

a) Using the chain rule, we have

$$\frac{d^2}{dx^2} = \frac{1}{x'}\frac{d}{dz}\left(\frac{1}{x'}\frac{d}{dz}\right) = \frac{1}{x'^2}\frac{d^2}{dz^2} - \frac{x''}{x'^3}\frac{d}{dz},$$

where $x' = dx/dz$. Thus the equation obeyed by $\psi(x(z))$ is

$$-\frac{1}{2}\left(\frac{d^2}{dz^2} - \frac{x''}{x'}\frac{d}{dz}\right)\psi + (x')^2[V(x(z)) - E]\psi = 0.$$

Following the recipe in the notes, we get rid of the first derivative term by setting $\psi = w(z)\tilde{\psi}$ where

$$w(z) = \exp\left\{-\frac{1}{2}\int^z \left(-\frac{x''(\zeta)}{x'(\zeta)}\right) d\zeta\right\} = \exp\left\{\frac{1}{2}\ln(x'(z))\right\} = \sqrt{x'(z)}.$$

We plug and chug to find that

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + (x')^2[V(z) - E] - \frac{1}{4}\left[\frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)\right]\right)\tilde{\psi} = 0.$$

b) It is not profitable to try to use the chain rule here. The algebra would be a nightmare. Instead observe that in the mapping diagram

$$\begin{array}{ccccc} & & z & & \\ & \nearrow & & \searrow & \\ x & & \longrightarrow & & w \end{array}$$

it cannot make any difference to the final equation were we change co-ordinates directly $x \to w$, or were we to map through the intermediate variable z. If we go directly we have

$$[V - E] \to \left(\frac{dx}{dw}\right)^2 [V - E] - \frac{1}{4}\{x, w\}.$$

If we proceed *via* z we end up with

$$[V - E] \to \left(\frac{dz}{dw}\right)^2 \left(\left(\frac{dx}{dz}\right)^2 [V - E] - \frac{1}{4}\{x, z\}\right) - \frac{1}{4}\{z, w\}.$$

These two expressions must be the same, and since

$$\left(\frac{dx}{dw}\right)^2 = \left(\frac{dz}{dw}\right)^2 \left(\frac{dx}{dz}\right)^2,$$

their equality necessitates only that

$$\{x, w\} = \left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\},$$

which is Cayley's identity.