Essential Calculus Review

We will make heavy use of the following interactions between derivatives and integrals:

a) Differentiating an integral with respect to the upper limit:

$$\frac{d}{db}\int_a^b f(y)\,dy = f(b).$$

b) Integrating a derivative

$$\int_a^b \frac{d}{dx} f(x)\,dx = f(b) - f(a).$$

c) Differentiating under the integral sign

$$\frac{d}{dx}\int_a^b f(y,x)\,dy = \int_a^b \frac{\partial}{\partial x} f(y,x)\,dy.$$

d) Leibniz' integral rule is a generalization of the last two equations:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \frac{db}{dt}f(b) - \frac{da}{dt}f(a) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx.$$

We can combine (a) with Leibniz' rule for derivatives

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx}$$

to get the integration-by-parts identity

$$f(b)g(b) - f(a)g(a) = \int_a^b \frac{df(x)}{dx}g(x)\,dx + \int_a^b f(x)\frac{dg(x)}{dx}\,dy.$$

This is often written as

$$[fg]_a^b - \int_a^b f'g\,dx = \int_a^b fg'\,dx.$$

Integration by parts is one of the most useful tools in calculus.