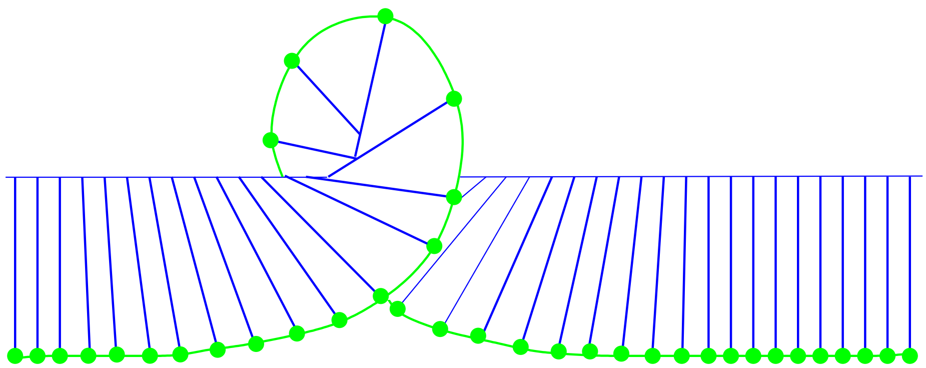
Physics 508 — **Mathematical Methods in Physics I** — Prof. M. Stone

Handout FN14 — `http://webusers.physics.uiuc.edu/~m-stone5` — 2117 ESB

Dec 15th 2014 — FINAL EXAM — University of Illinois

This exam has **four** *pages and* **five** *problems. Each problem is worth 20 points. Do as many as you can. Errors will not be propagated, so make sure of each step before you go on.*

1) Soliton twist: A large number of unit length pendulums are suspended from a common axis and coupled together by some elastic material to form a ribbon. A single twist is then put in the ribbon as shown in the figure. (Gravity is acting downwards.)



Coupled pendulums forming a ribbon.

We treat the ribbon as continuous, so the potential-energy functional for the ribbon can be written as

$$V[\theta] = \int_{-\infty}^{\infty} \left\{ \frac{\kappa}{2} \left(\frac{\partial \theta}{\partial x} \right)^2 + m(1 - \cos\theta) \right\} dx.$$

Here $\theta(x)$ is the angle that the pendulum situated at x is making with the vertical. The parameter κ is a spring constant, and $m\,dx$ is the total weight (mass times g) of the pendulum bobs in the interval dx . We wish to find the (time independent) $\theta(x)$ that minimizes $V[\theta]$ subject to the boundary conditions $\theta(-\infty) = 0$ and $\theta(+\infty) = 2\pi$.

a) Use the calculus of variations to find the equation that determines the minimum-potential-energy configuration. [5 points]

b) Solve the equation you found in part (a). (A first integral is useful.) Your $\theta(x)$ will be of the form

$$\theta(x) = A \tan^{-1}\{\exp B(x - x_0)\},$$

where you should find explicit expressions for A, B. [15 points]

Useful:

$$(1 - \cos 2x) = 2\sin^2 x, \quad \frac{d}{dx} \ln \tan(x/2) = \frac{1}{\sin x}.$$

2) Green function and Fredholm: Seek a solution to the equation

$$-\frac{d^2y}{dx^2} = f(x), \quad x \in [0,1]$$

with inhomogeneous boundary conditions $y'(0) = F_0$, $y'(1) = F_1$. Observe that the corresponding homogeneous boundary condition problem has a zero mode. Therefore the solution, if one exists, cannot be unique.

a) Show that there can be no solution to the differential equation and inhomogeneous boundary condition unless $f(x)$ satisfies the condition

$$\int_0^1 f(x)\,dx = F_0 - F_1. \quad (\star)$$

[5 points]

b) Let $G(x,\xi)$ denote the modified Green function

$$G(x,\xi) = \begin{cases} \frac{1}{3} - \xi + \frac{x^2+\xi^2}{2}, & 0 < x < \xi \\ \frac{1}{3} - x + \frac{x^2+\xi^2}{2}, & \xi < x < 1, \end{cases}.$$

Use the Lagrange-identity method for inhomogeneous boundary conditions to deduce that if a solution exists then it necessarily obeys

$$y(x) = \int_0^1 y(\xi)\,d\xi + \int_0^1 G(\xi,x) f(\xi)\,d\xi + G(1,x)F_1 - G(0,x)F_0.$$

[5 points]

c) By differentiating with respect to x, show that

$$y_{\text{tentative}}(x) = \int_0^1 G(\xi,x) f(\xi)\,d\xi + G(1,x)F_1 - G(0,x)F_0 + C,$$

where C is an arbitrary constant, obeys the boundary conditions. [5 points]

d) By differentiating a second time with respect to x, show that $y_{\text{tentative}}(x)$ is a solution of the differential equation if, and only if, the condition $\star$ is satisfied. [5 points]

3) Hankel Transforms: The orthogonality relation for the Bessel functions $J_0(kx)$, $k \in [0, \infty)$, on the positive real line is

$$\int_0^\infty J_0(k_1 x) J_0(k_2 x) x \, dx = \frac{1}{k_1} \delta(k_1 - k_2).$$

a) Write down the corresponding *completeness relation.* [5 points]

b) Given that

$$\int_0^\infty e^{-ax} J_0(kx) \, dx = \frac{1}{\sqrt{k^2 + a^2}},$$

evaluate

$$\int_0^\infty \frac{J_0(kx)}{\sqrt{x^2 + a^2}} x \, dx. \qquad \text{[5 points]}$$

c) By using the definition

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} e^{ix \sin\theta} \, d\theta,$$

and by computing the two-dimensional Fourier transform of

$$F(x, y) = \exp\left\{-\frac{a}{2}(x^2 + y^2)\right\},$$

evaluate

$$\int_0^\infty J_0(kx) e^{-ax^2/2} x \, dx. \qquad \text{[10 points]}$$

Useful:

$$\int_{-\infty}^\infty \exp\left\{-\frac{a}{2}x^2\right\} e^{ikx} \, dx = \sqrt{\frac{2\pi}{a}} \exp\left\{-\frac{1}{2a}k^2\right\}.$$

4) Legendre Polynomial Normalization: The Legendre polynomials $P_n(x)$ may be defined by their generating function

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^\infty t^n P_n(x).$$

You may assume that we already know that

$$\int_{-1}^1 P_n(x) P_{n'}(x) \, dx = 0, \quad n \neq n'.$$

a) Evaluate the integral

$$\int_{-1}^1 \frac{1}{1 - 2tx + t^2} \, dx. \qquad \text{[10 points]}$$

b) By expanding your result from part (a) as a power series in t and examining the coefficient of t^{2n}, evaluate

$$\int_{-1}^{1} [P_n(x)]^2 \, dx$$

as a rational function of n. [10 points]

Hint: You may find the (hopefully known to you!) series expansion

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

to be of use.

5) Integral Equations:

a) Solve the integral equation

$$u(x) = f(x) + \lambda \int_0^1 x^3 y^3 u(y) dy, \quad 0 < x < 1$$

for the unknown $u(x)$ in terms of the given function $f(x)$. For what values of λ does a unique solution $u(x)$ exist without restrictions on $f(x)$? For what value $\lambda = \lambda_0$ does a solution exist only if $f(x)$ satisfies some condition? Using the language of the Fredholm alternative, and the range and nullspace of the relevant operators, explain what is happening when $\lambda = \lambda_0$. For the case $\lambda = \lambda_0$ find explicitly the condition on $f(x)$ and, assuming this condition is satisfied, write down the corresponding general solution for $u(x)$. Check that this solution does indeed satisfy the integral equation. [10 points].

b) Use a Laplace transform to find the solution $u(x)$ to the generalized Abel equation

$$f(x) = \int_0^x (x-t)^{-\mu} \, u(t) dt, \quad 0 < \mu < 1,$$

where $f(x)$ is given and $f(0) = 0$. Your solution will be of the form

$$u(x) = \int_0^x K(x-t) f'(t) dt.$$

where you should give an explicit expression for the kernel $K(x-t)$. [10 points]

(Useful Formula: $\displaystyle\int_0^\infty t^{\mu-1} e^{-pt} dt = p^{-\mu} \, \Gamma(\mu), \quad \mu > 0.)$