Mathematical Methods in Physics I

HOMEWORK 1

Revision and Warm-up exercises

These problems are designed to reactivate your mathematical skills after a relaxing summer break. They are not designed to be easy! Each one has some twist that is designed to catch you out if you merely manipulate without thinking.

Differential Calculus: After taking a previous version of this course, a student claimed that the expression

$$y(x) = \frac{1}{\omega \sin \omega L}\left\{\int_0^x f(t)\sin\omega(x-L)\sin\omega t\,dt + \int_x^L f(t)\sin\omega x\sin\omega(t-L)\,dt\right\}$$

is the solution to the problem: "Find $y(x)$ obeying the differential equation

$$\frac{d^2y}{dx^2} + \omega^2 y = f(x)$$

on the interval $[0, L]$ and satisfying the boundary conditions $y(0) = 0 = y(L)$." First examine her solution to see if it obeys the boundary conditions. Then, by differentiating her solution twice with respect to x and substituting the result into the differential equation, investigate whether her solution is correct.

A second student claimed that if $K(x)$ and $f(x)$ are smooth functions with $K(0) = 1$, and we set

$$F(x) = \int_0^x K(x-y)f(y)\,dy,$$

then

$$F'(x) = \int_0^x K(x-y)f'(y)\,dy.$$

(Here the "prime" denotes differentiation with respect to x or y, as appropriate.) Was he right? If so, explain why. If not, find an extra condition on $f(x)$ that makes his equation correct.

Improper Integrals: Let $\mu > \lambda > 0$ be real numbers. Sketch by hand a graph of the function

$$F(t) = \frac{e^{-\lambda t} - e^{-\mu t}}{t}, \quad 0 < t < \infty.$$

Now a student wishes to evaluate the integral

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-\lambda t} - e^{-\mu t}}{t}\,dt.$$

He breaks it up as

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-\lambda t}}{t}\,dt - \int_0^\infty \frac{e^{-\mu t}}{t}\,dt.$$

In the first integral he makes the substitution $x = \lambda t$. In the second he sets $x = \mu t$. He ends up with

$$I(\lambda, \mu) = \int_0^\infty \frac{e^{-x}}{x}\, dx - \int_0^\infty \frac{e^{-x}}{x}\, dx.$$

As the two integrals are identical, he concludes that they must cancel and so $I(\lambda, \mu) = 0$. From your sketch of the function being integrated, you know that he has gone wrong somewhere. Locate the error in his method, and make the small but crucial modification to the lower limit that leads to the correct answer. Confirm your result by using Feynman's trick of first computing the easy integral for $\partial I / \partial \mu$, and then integrating the result with respect to μ.

Now use the same "small modification" technique at *both* ends of the integration interval, and so easily evaluate the intimidating-looking integral

$$I = \int_0^\infty \ln \left\{ \frac{a + be^{-px}}{a + be^{-qx}} \right\} \frac{dx}{x}.$$

Assume that a, b, p, q are positive real numbers.

Trigonometric functions: Imagine that when solving a one-dimensional quantum mechanics scattering problem you find a wavefunction $\psi(x) = \sin(kx + \delta)$ where you know that the phase shift δ is a smooth (infinitely differentiable) function of k that obeys the equation

$$\cot(ka + \delta) - \cot ka = \lambda.$$

Solving formally for $\delta(k)$, you write

$$\delta(k) = -ka + \cot^{-1}(\lambda + \cot ka).$$

First assume that $\lambda = 0$ and draw the graph of $\delta(k)$ versus k. Now consider effect the on your graph of a nonzero and positive λ. If λ is fairly large compared to unity, sketch *by hand* the graph of the function $\delta(k)$. Label the points on the k axis where ka is a multiple of π, so I can see their relation to the places where $\delta(k)$ has wiggles.

The point of this problem is to understand what the graph looks like without using a computer or graphing calculator. Indeed programs such as *Mathematica*™ fail to plot this function correctly. Your solution should therefore include the reasoning that lead to your curve.

Integration by Parts: As part of a project a graduate student is given a function $y(x)$ and needs to solve the integral equation

$$y(x) = \chi(x) - \int_0^x \frac{D'(\xi)}{D(\xi)} \chi(\xi)\, d\xi, \qquad (\star)$$

and so find the function $\chi(x)$. Here $D(x)$ is a smooth real-valued function defined on $[0, L]$ that is never equal to zero, and $D'(\xi)$ denotes $dD/d\xi$. The given function $y(x)$ is a smooth, real-valued function defined on $[0, L]$ such that $y(0) = 0$. Her project advisor suggests that

$$\chi(x) = y(x) + D(x) \int_0^x \frac{D'(\xi)}{D^2(\xi)} y(\xi)\, d\xi. \quad (\star\star)$$

might work. Plug the suggested formula $(\star\star)$ for $\chi(x)$ into the right-hand side of $(\star)$ and use integration by parts to see if it does indeed solve the equation.

Her advisor also conjectures that a solution $\chi(x)$ not only *exists* for every $y(x)$ but is also *uniquely determined* by $y(x)$. To explore this possibility, plug the representation of $y(x)$ given by the right-hand side of $(\star)$ into the right-hand side of $(\star\star)$ and again use integration by parts to see if it outputs the $\chi(x)$ that appears in $(\star)$. If so, use this fact to establish uniqueness. (**Hint**: $(D'/D^2) = -(1/D)'$.)

Matrix Algebra: Let V be an N-dimensional vector space, where $N > 1$. Consider a linear operator $T : V \to V$ which, in some basis, is represented by an $N \times N$ matrix $\mathbf{T}$. The operator T obeys the equation

$$(T - \lambda I)^p = 0,$$

with $p = N$, but does not obey this equation for any integer $p < N$. Here λ is a real number and I is the identity operator.

i) Show that if T possesses an eigenvector, the corresponding eigenvalue must be λ. Deduce that T *cannot* be diagonalized.

ii) Show that there exists a vector $\mathbf{e}_1$ such that $(T - \lambda I)^N \mathbf{e}_1 = 0$, but no lesser power of $(T - \lambda I)$ kills $\mathbf{e}_1$.

iii) Define $\mathbf{e}_2 = (T - \lambda I)\mathbf{e}_1$, $\mathbf{e}_3 = (T - \lambda I)^2 \mathbf{e}_1$, *etc.* up to $\mathbf{e}_N$. Show that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_N$ are linearly independent.

iv) Use $\mathbf{e}_1, \ldots, \mathbf{e}_N$ as a basis for your vector space. Taking

$$\mathbf{e}_1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_N = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

write out the matrix $\mathbf{T}$ representing T in the $\mathbf{e}_i$ basis.