Physics 508 | | Prof. M. Stone
Handout 12 | **Mathematical Methods in Physics I** | 2117 ESB
Fall 2025 | HOMEWORK 12 | University of Illinois

Here are some optional problems on integral equations. They are taken *verbatim* from Paul Goldbart's homework sets.

1) Integral equations:

a) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = \mathbf{e}^x + \lambda \int_0^1 xy\, u(y)\, dy\,.$$

b) Solve the homogeneous type II Fredholm integral equation

$$u(x) = \lambda \int_0^\pi \sin(x-y)\, u(y)\, dy\,.$$

c) Solve the inhomogeneous type II Fredholm integral equation

$$u(x) = x + \lambda \int_0^1 y(x+y)\, u(y)\, dy$$

to second order in λ using

i) the Liouville-Neumann-Born series; and

ii) the Fredholm series.

d) By differentiating, solve the integral equation: $u(x) = x + \int_0^x u(y)\, dy$.

e) Solve the integral equation: $u(x) = x^2 + \int_0^1 xy\, u(y)\, dy$.

f) Find the eigenfunction(s) and eigenvalue(s) of the integral equation

$$u(x) = \lambda \int_0^1 \mathbf{e}^{x-y}\, u(y)\, dy\,.$$

g) Solve the integral equation: $u(x) = \mathbf{e}^x + \lambda \int_0^1 \mathbf{e}^{x-y}\, u(y)\, dy$.

2) Neumann Series: Consider the integral equation

$$u(x) = g(x) + \lambda \int_0^1 K(x, y)\, u(y)\, dy\,,$$

in which only u is considered unknown.

a) Write down the solution $u(x)$ to second order in the Liouville-Neumann-Born series.

b) Suppose $g(x) = x$ and $K(x, y) = \sin 2\pi xy$. Compute $u(x)$ to second order in the Liouville-Neumann-Born series. (You may leave your answer to the second-order term in terms of a single integral.)

3) Translationally invariant kernels:

a) Consider the integral equation: $u(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y)\, u(y)\, dy$, with the translationally invariant kernel $K(x, y) = Q(x-y)$, in which g, λ and Q are considered known. Show that the Fourier transforms $\hat{u}$, $\hat{g}$ and $\hat{Q}$ satisfy $\hat{u}(q) = \hat{g}(q)/\{1 - \sqrt{2\pi}\lambda\hat{Q}(q)\}$. Expand this result to second order in λ to recover the second-order Liouville-Neumann-Born series.

b) Use Fourier transforms to find a solution of the integral equation

$$u(x) = \mathbf{e}^{-|x|} + \lambda \int_{-\infty}^{\infty} \mathbf{e}^{-|x-y|}\, u(y)\, dy$$

which remains finite as $|x| \to \infty$.

c) Use Laplace transforms to find a solution for $x > 0$ of the integral equation

$$u(x) = \mathbf{e}^{-x} + \lambda \int_0^x \mathbf{e}^{-|x-y|}\, u(y)\, dy\,.$$