Physics 508 Prof. M. Stone
Handout 4 **Mathematical Methods in Physics I** 2117 ESB
Fall 2025 HOMEWORK 4 University of Illinois

1) Test functions and distributions:

a) Let $f(x)$ be a smooth function.

i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x).$$

ii) We might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$

By integrating both against a test function, show this new expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).

b) In a paper[1] that has recently been cited in the literature on topological insulators a distribution $\delta^{(1/2)}(x)$ is defined by setting

$$\delta^{(1/2)}(x) \text{``}=\text{''} \int_{-\infty}^{\infty} \frac{dk}{2\pi} |k|^{1/2} e^{ikx}.$$

The scare quotes "…" around the equal sign are there because the Fourier transform on the RHS is clearly divergent. We therefore need to decide how to interpret it. Let's try to define the evaluation of $\delta^{(1/2)}$ on a test function $\varphi(x)$ as

$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x)\, dx \stackrel{\text{def}}{=} \lim_{\mu\to 0_+} \left\{ \int_{-\infty}^{\infty} \delta_\mu^{(1/2)}(x)\varphi(x)\, dx \right\}.$$

where

$$\begin{aligned} \delta_\mu^{(1/2)}(x) &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{ikx}|k|^{1/2} e^{-\mu|k|} \frac{dk}{2\pi} \\ &= \sqrt{\frac{1}{4\pi}} (x^2+\mu^2)^{-3/4} \cos\left(\frac{3}{2}\tan^{-1}\left(\frac{x}{\mu}\right)\right). \end{aligned}$$

(Could you have evaluated this integral if I had not given you the answer?)
Plot some graphs of $\delta_\mu^{(1/2)}(x)$ for various values of μ, and so get an idea of how it behaves as the convergence factor $e^{-\mu|k|} \to 1$. Deduce that

$$\int_{-\infty}^{\infty} \delta^{(1/2)}(x)\varphi(x)\, dx = -\sqrt{\frac{1}{8\pi}} \int_{-\infty}^{\infty} \frac{1}{|x|^{3/2}} \{\varphi(x) - \varphi(0)\}\, dx.$$

(Hint: Observe that $\delta_\mu^{(1/2)}(x)$ is the Fourier transform of a function that vanishes at $k=0$. What property of the the graph of $\delta_\mu^{(1/2)}(x)$ does this imply?)

[1] H. Aratyn, *Fermions from Bosons in 2+1 dimensions*, Phys. Rev. D **28** (1983) 2016-18.

c) Let $\varphi(x)$ be a test function. Using the definition of the *principal part integral*, show that

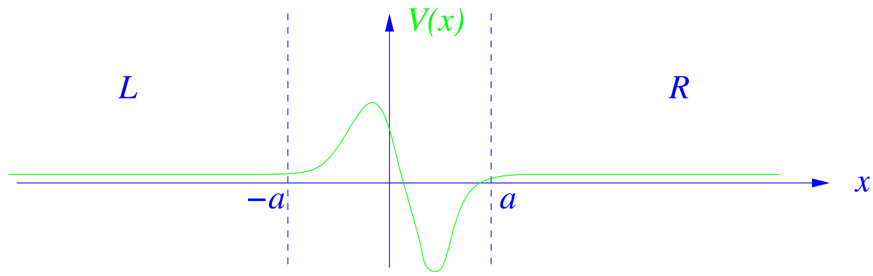
$$\frac{d}{dt}\left\{P\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x-t)}\,dx\right\}=P\int_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,dx$$

To do this fix the value of the cutoff ϵ and then differentiate the resulting ϵ-regulated integral, taking care to include the terms arising from the t dependence of the limits at $x = t \pm \epsilon$.

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation

$$-\frac{d^2\psi}{dx^2}+V(x)\psi=E\psi,\quad V(x)\in\mathbb{R},$$

where $V(x)$ is zero except in a finite interval $[-a, a]$ near the origin.



Let L denote the left asymptotic region, $-\infty < x < -a$, and similarly let R denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$\psi_k(x)=\begin{cases} e^{ikx}+r_L(k)e^{-ikx}, & x\in L,\\ t_L(k)e^{ikx}, & x\in R,\end{cases}$$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$\psi_k(x)=\begin{cases} t_R(k)e^{ikx}, & x\in L,\\ e^{ikx}+r_R(k)e^{-ikx}, & x\in R.\end{cases}$$

The wavefunctions in $[-a, a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) $|r_{L,R}|^2 + |t_{L,R}|^2 = 1$,
b) $t_L(k)=t_R(-k)$.
c) Deduce from parts a) and b) that $|r_L(k)| = |r_R(-k)|$.
d) Take the specific example of $V(x) = \lambda\delta(x-b)$ with $|b| < a$. Compute the transmission and reflection coefficients and hence show that $r_L(k)$ and $r_R(-k)$ may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that $y_1 = u(x)$ is one solution to the equation

$$y'' + V(x)y = 0.$$

By trying $y = u(x)v(x)$ show that

$$y_2 = u(x)\int^x \frac{d\xi}{u^2(\xi)}$$

is also a solution of the differential equation. Is this new solution ever merely a constant mutiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian $W(y_2, y_1)$.)

b) Suppose that we are told that the product, $y_1 y_2$, of the two solutions to the equation $y'' + p_1 y' + p_2 y = 0$ is a constant. Show that this requires $2p_1 p_2 + p_2' = 0$.

c) By using ideas from part b) or otherwise, find the general solution of the equation

$$(x+1)x^2 y'' + xy' - (x+1)^3 y = 0.$$

4) Normal forms and the Schwarzian derivative: We saw in class that if y obeys a second-order linear differential equation

$$y'' + p_1 y' + p_2 y = 0$$

then we can make always make a substitution $y = w\tilde{y}$ so that $\tilde{y}$ obeys an equation without a first derivative:

$$\tilde{y}'' + q(x)\tilde{y} = 0.$$

Suppose $\psi(x)$ obeys a Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + [V(x) - E]\right)\psi = 0.$$

a) Make a smooth and invertible change of independent variable by setting $x = x(z)$ and find the second order differential equation in z obeyed by $\psi(z) \equiv \psi(x(z))$. Find the $\tilde{\psi}(z)$ that obeys an equation with no first derivative. Show that this equation is

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\}\right)\tilde{\psi}(z) = 0,$$

where the primes denote differentiation with respect to z, and

$$\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

is called the *Schwarzian* derivative of x with respect to z. Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of maps $x \to z \to w$ to establish *Cayley's identity*

$$\left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\} = \{x, w\}.$$

(Hint: If this takes you more than a line or two, or you find yourself using the hideous expression for $\{x, z\}$, you are missing the point of the problem.)