Physics 508 — Prof. M. Stone
Handout 4 — **Mathematical Methods in Physics I** — 2117 ESB
Fall 2024 — Homework 5 — University of Illinois

1) Linear differential operators:

a) Let $w(x) > 0$. Consider the differential operator $\hat{L} = id/dx$. Find the formal adjoint of L with respect to the inner product $\langle u|v\rangle_w = \int wu^*v\,dx$, and find the corresponding surface term $Q[u,v]$.

b) Now do the same for the operator $M = d^4/dx^4$, for the case $w = 1$. Find the adjoint boundary conditions defining the domain of $M^\dagger$ for the case

$$\mathcal{D}(M) = \{y, y^{(4)} \in L^2[0,1] : y(0) = y'''(0) = y(1) = y'''(1) = 0\}.$$

(Hint: you may find the identity

$$f^{(4)}g - fg^{(4)} = \frac{d}{dx}\{f'''g - f''g' + f'g'' - fg'''\}$$

to be of use.)

2) Sturm-Liouville forms: By constructing appropriate weight functions convert the following common operators into Sturm-Liouville form:

a) $\hat{L} = (1-x^2)\,d^2/dx^2 + [(\mu-\nu) - (\mu+\nu+2)x]\,d/dx$.
b) $\hat{L} = (1-x^2)\,d^2/dx^2 - 3x\,d/dx$.
c) $\hat{L} = d^2/dx^2 - 2x(1-x^2)^{-1}\,d/dx - m^2\,(1-x^2)^{-1}$.

3) Discrete approximations and self-adjointness: Consider the second order inhomogeneous equation $Lu \equiv u'' = g(x)$ on the interval $0 \le x \le 1$. Here $g(x)$ is known and $u(x)$ is to be found. We wish to solve the problem on a computer, and so set up a discrete approximation to the ODE in the following way:

- replace the continuum of independent variables $0 \le x \le 1$ by the discrete lattice of points $0 \le x_n \equiv n/N \le 1$ Here N is a positive integer and $n = 0, 1, 2, \ldots, N$;
- replace the functions $u(x)$ and $g(x)$ by the arrays of real variables $u_n \equiv u(x_n)$ and $g_n \equiv g(x_n)$;
- approximate the continuum differential operator d^2/dx^2 by the finite difference operator $\mathcal{D}^2$, defined by $\mathcal{D}^2 u_n \equiv (u_{n+1} - 2u_n + u_{n-1})/a^2$ where $a = N^{-1}$ is the lattice spacing.

Now do the following problems:

a) Impose continuum Dirichlet boundary conditions $u(0) = u(1) = 0$. Decide what these correspond to in the discrete approximation, and write the resulting set of algebraic equations in matrix form. Show that the corresponding matrix is real and symmetric.

b) Impose the periodic boundary conditions $u(0) = u(1)$ and $u'(0) = u'(1)$, and show that these require us to set $u_0 \equiv u_N$ and $u_{N+1} \equiv u_1$. Again write the system of algebraic equations in matrix form and show that the resulting matrix is real and symmetric.

c) Consider the non-symmetric $N \times N$ matrix operator

$$D^2 u = \frac{1}{a^2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 1 & -2 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_N \\ u_{N-1} \\ u_{N-2} \\ \vdots \\ u_3 \\ u_2 \\ u_1 \end{pmatrix}.$$

i) What vectors span the null space of D^2?

ii) To what continuum boundary conditions for d^2/dx^2 does this matrix correspond?

iii) Consider the matrix $(D^2)^\dagger$, To what continuum boundary conditions does this matrix correspond? Are they the adjoint boundary conditions for the operator in part ii)?

4) Factorization: Schrödinger equations of the form

$$-\frac{d^2\psi}{dx^2} - l(l+1)\text{sech}^2 x\, \psi = E\psi$$

are known as *Pöschel-Teller equations*. By setting $u = l \tanh x$ and following the strategy of this problem one may relate solutions for l to those for $l-1$ and so find all bound states and scattering eigenfunctions for any integer l.

a) Suppose that we know that $\psi = \exp\{-\int^x u(x')dx'\}$ is a solution of

$$L\psi \equiv \left(-\frac{d^2}{dx^2} + W(x)\right)\psi = 0.$$

Show that L can be written as $L = M^\dagger M$ where

$$M = \left(\frac{d}{dx} + u(x)\right), \quad M^\dagger = \left(-\frac{d}{dx} + u(x)\right),$$

the adjoint being taken with respect to the product $\langle u|v\rangle = \int u^* v\, dx$.

b) Now assume L is acting on functions on $[-\infty, \infty]$ and that we not have to worry about boundary conditions. Show that given an eigenfunction ψ_- obeying $M^\dagger M \psi_- = \lambda \psi_-$ we can multiply this equation on the left by M and so find a eigenfunction ψ_+ with the same eigenvalue for the differential operator

$$L' = MM^\dagger = \left(\frac{d}{dx} + u(x)\right)\left(-\frac{d}{dx} + u(x)\right)$$

and *vice-versa*. Show that this correspondence $\psi_- \leftrightarrow \psi_+$ will fail if, *and only if*, $\lambda = 0$.

c) Apply the strategy from part b) in the case $u(x) = \tanh x$ and one of the two differential operators $M^\dagger M$, $MM^\dagger$ is (up to an additive constant)

$$H = -\frac{d}{dx}^2 - 2\,\text{sech}^2 x.$$

Show that H has eigenfunctions of the form $\psi_k = e^{ikx} P(\tanh x)$ and eigenvalue $E = k^2$ for any k in the range $-\infty < k < \infty$. The function $P(\tanh x)$ is a polynomial in $\tanh x$ which you should be able to find explicitly. By thinking about the exceptional case $\lambda = 0$, show that H has an eigenfunction $\psi_0(x)$, with eigenvalue $E = -1$, that tends rapidly to zero as $x \to \pm\infty$. Observe that there is no corresponding eigenfunction for the other operator of the pair.