

**1) Missing State:** In Homework Set 4 you found that the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} - 2 \operatorname{sech}^2 x\right) \psi = E \psi$$

has eigensolutions

$$\psi_k(x) = e^{ikx}(-ik + \tanh x)$$

with eigenvalue  $E = k^2$ .

- Show that for  $x$  large and positive  $\psi_k(x) \approx A e^{ikx} e^{i\delta(k)}$ , while for  $x$  large and negative  $\psi_k(x) \approx A e^{ikx} e^{-i\delta(k)}$ , the complex constant  $A$  (which you must write down explicitly) being the same in both cases. Express  $\delta(k)$  as the inverse tangent of an algebraic expression in  $k$ .
- Impose periodic boundary conditions  $\psi(-L/2) = \psi(+L/2)$  where  $L \gg 1$ . Find the allowed values of  $k$  and hence an explicit expression for the  $k$ -space density,  $\rho(k) = \frac{dn}{dk}$ , of the eigenstates.
- Compare your formula for  $\rho(k)$  with the corresponding expression,  $\rho_0(k) = L/2\pi$ , for the eigenstate density of the zero-potential equation and compute the integral

$$\Delta N = \int_{-\infty}^{\infty} \{\rho(k) - \rho_0(k)\} dk.$$

- Deduce that one eigenfunction has gone missing from the continuum and presumably become a localized bound state. (You will have found an explicit expression for this localized eigenstate in Homework Set 4.)

**2) Continuum Completeness:** Consider the differential operator

$$\hat{L} = -\frac{d^2}{dx^2}, \quad 0 \leq x < \infty$$

with self-adjoint boundary conditions  $\psi(0)/\psi'(0) = \tan \theta$  for some fixed angle  $\theta$ .

- Show that when  $\tan \theta < 0$  there is a single normalizable negative-eigenvalue eigenfunction localized near the origin, but none when  $\tan \theta > 0$ .
- Show that there is a continuum of positive-eigenvalue eigenfunctions of the form  $\psi_k(x) = \sin(kx + \delta(k))$  where the phase shift  $\delta$  is found from

$$e^{i\delta(k)} = \frac{1 + ik \tan \theta}{\sqrt{1 + k^2 \tan^2 \theta}}.$$

- Write down (no justification required) the appropriate completeness relation

$$\delta(x - x') = \int \frac{dn}{dk} N_k \psi_k(x) \psi_k(x') dk + \sum_{\text{bound}} \psi_n(x) \psi_n(x')$$

with an explicit expression for the product (not the separate factors) of the density of states and the normalization constant  $N_k$ , and with the correct limits on the integral over  $k$ .

- Confirm that the  $\psi_k$  continuum on its own, or together with the bound state when it exists, form a complete set. You will do this by evaluating the integral

$$I(x, x') = \frac{2}{\pi} \int_0^\infty \sin(kx + \delta(k)) \sin(kx' + \delta(k)) dk$$

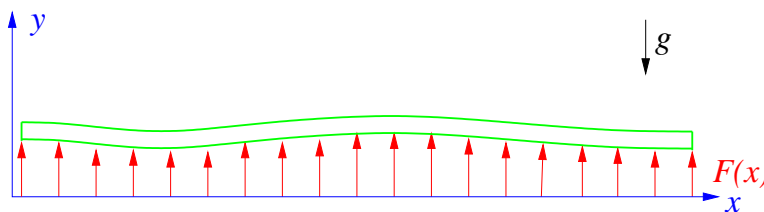
and interpreting the result. You will need the following standard integral

$$\int_{-\infty}^\infty \frac{dk}{2\pi} e^{ikx} \frac{1}{1 + k^2 t^2} = \frac{1}{2|t|} e^{-|x|/|t|}.$$

To get full credit, you must show how the bound state contribution switches on and off with  $\theta$ . The modulus signs are essential for this.

### 3) Fredholm Alternative:

A heavy elastic bar with uniform mass  $m$  per unit length lies almost horizontally. It is supported by a distribution of upward forces  $F(x)$ .



The shape of the bar,  $y(x)$ , can be found by minimizing the energy

$$U[y] = \int_0^L \left\{ \frac{1}{2} \kappa (y'')^2 - (F(x) - mg)y \right\} dx,$$

which gives (homework 2!) the equation

$$\hat{L}y \equiv \kappa \frac{d^4 y}{dx^4} = F(x) - mg, \quad y'' = y''' = 0 \quad \text{at} \quad x = 0, L.$$

- Show that the boundary conditions are such that the operator  $\hat{L}$  is self-adjoint with respect to an inner product with weight function 1.
- Find the zero modes which span the null space of  $\hat{L}$ .
- If there are  $n$  linearly independent zero modes, then the codimension of the range of  $\hat{L}$  is also  $n$ . Using your explicit solutions from the previous part, find the conditions that must be obeyed by  $F(x)$  for a solution of  $\hat{L}y = F - mg$  to exist. What is the physical meaning of these conditions?
- The solution to the equation and boundary conditions is not unique. Is this non-uniqueness physically reasonable? Explain.