Physics 498/MMA — **Mathematical Methods in Physics I** — Prof. M. Stone
Handout FT — `http://w3.physics.uiuc.edu/~m-stone5` — 2117 ESB
Dec 20th 2002 — FINAL EXAM — University of Illinois

This exam has **four** *pages and* **six** *problems. Answer question* **one**, *and then any* **other three** *questions.* **Do not hand in solutions to more than this number of problems!** *Try to answer entire questions. Little, if any, credit will be given for fragmentary answers. Errors will not be propagated, so make sure of each step before you go on.*

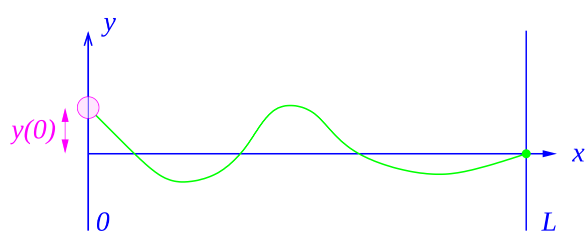
1) One-dimensional Green Function: Consider the differential equation

$$-y'' = f(x)$$

with boundary conditions $y'(0) = 0$ and $y(1) = 0$. We are given $f(x)$ and wish to solve for $y(x)$.

a) Construct the explicit Green function appropriate to this problem (5 points).
b) Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the unit interval (5 points).
c) Evaluate the x derivative of your solution, $y(x)$, and confirm that y obeys both boundary conditions (5 points).
d) Take one further derivative of your $y(x)$ and confirm that it does indeed solve the original problem (5 points).

2) Bead and string: A bead of mass M is free to slide up and down the y axis.



A bead connected to a string.

It is attached to the $x = 0$ end of a string in such a way that the Lagrangian for the string-bead system is

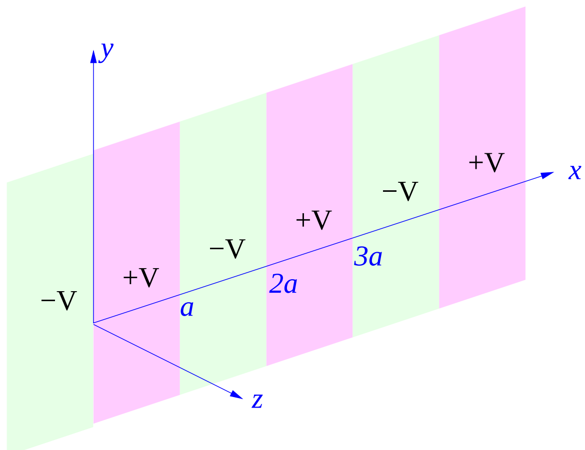
$$L = \frac{1}{2}M[\dot{y}(0)]^2 + \int_0^L \left\{ \frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2 \right\} dx.$$

Here ρ is mass per unit length of the string and T its tension.

a) Write down the Euler-Lagrange equations which come from varying $y(x)$ subject to the condition that $y(L)$ is fixed. You should end up with an equation for the string in the interval $0 < x \leq 0$, and another governing the motion of the bead. (10 points)

b) Write down the *First Integral* appropriate to your system of Euler-Lagrange equations. Verify, from your equations of motion for the bead and string, that your expression for the first integral is independent of time. (10 points)

3) Alternating stripes: The plane $z = 0$ is occuped by infinitely long conducting strips of width a.



Infinite stripes.

The infinite strip $0 < x < a$, $-\infty < y < \infty$ is held at potential V. The strip $a < x < 2a$, $-\infty < y < \infty$ is held at potential $-V$, and this pattern is repeated periodically to cover the entire $z = 0$ plane.

a) Express the given potential as a Fourier series of the form

$$V(x) = \sum_0^\infty \left(a_n \sin \frac{2\pi n x}{2a} + b_n \cos \frac{2\pi n x}{2a} \right),$$

and find an **explicit expression** for the coefficients a_n, b_n. (**Hint**: Only one of the sine or cosine series is actually needed.) [6 points]

b) Given that the potential $V(x, y, z)$ tends to zero as $|z| \to \infty$, adopt the series from part a) to find $V(x, y, z)$ at all points in $\mathbf{R}^3$ [8 points].

c) Using $\mathbf{E} = -\nabla V$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, write down the series giving the surface charge $\sigma(x)$. Perform the sum to find an explict formula for $\sigma(x)$. **Hint**: You may need to include a convergence factor. The answer is an elementary trigonometric function. (It will help to sketch a graph of $\sigma(x)$ to make sure that your solution makes physical sense.) [6 points]

4) Factoring Bessel: Consider Bessel's equation

$$y'' + \frac{1}{x}y' + \left(k^2 - \frac{\nu^2}{x^2}\right) y = 0.$$

a) Write Bessel's equation as an eigenvalue problem $L_\nu y = k^2 y$ and factorize the operator L in the form

$$L_\nu = \left(\pm\frac{d}{dx} + \frac{\alpha}{x}\right)\left(\mp\frac{d}{dx} + \frac{\beta}{x}\right).$$

You should be able to find *two* inequivalent factorizations (5 points).

b) Show that Bessel's equation can be written in the generalized Sturm-Liouville form

$$-\frac{1}{\mu(x)}\frac{d}{dx}p(x)\frac{d}{dx}y + \frac{\nu^2}{x^2}y = k^2 y.$$

Explain clearly how this form determines the measure in the inner product with respect to which the Bessel operator is formally self-adjoint (5 points).

c) Find the formal adjoint $D^\dagger$ of the operator

$$D = \frac{d}{dx}$$

with respect to the inner product you found in part b) (5 points).

d) Using your result from part c), show that your factorizations from part a) can be written $L_\nu = A_\nu^\dagger A_\nu = A_{\nu-1}A_{\nu-1}^\dagger$ for suitable operator A_ν, Briefly sketch how this and the results of parts a)—c) allow us to deduce recurrence relations determining $J_{\nu+1}(kx)$ in terms of $J_\nu(kx)$ and its derivative (5 points).

5) Dirichlet Problem: Write a short but clear account of how knowledge of the Green function $G(x, x')$ with

$$\begin{aligned} -\nabla_x^2 G(x,x') &= \delta^d(x-x'), \quad x \in \Omega, \\ G(x,x') &= 0, \quad x \in \partial\Omega, \end{aligned}$$

allows us to solve the Dirichlet problem

$$\begin{aligned} -\nabla^2\varphi &= f(x), \quad x \in \Omega, \\ \varphi(x) &= F(x), \quad x \in \partial\Omega, \end{aligned}$$

for a bounded and simply connected domain $\Omega \subset \mathbf{R}^d$. To receive full credit your account should include a proof that $G(x,x') = G(x',x)$, and an explicit formula expressing $\varphi(x)$ as a sum of integrals over Ω and its boundary $\partial\Omega$ (20 points).

6) Integral Equations:

a) Solve the integral equation

$$u(x) = f(x) + \lambda \int_0^1 x^2 y^2 u(y) dy, \quad 0 < x < 1$$

for the unknown $u(x)$ in terms of the given function $f(x)$. For what values of λ does a unique solution $u(x)$ exist without restrictions on $f(x)$? For what value $\lambda = \lambda_0$ does a solution exist only if $f(x)$ satisfies some condition? Using the language of the Fredholm alternative, and the range and nullspace of the relevant operators, explain what is happening when $\lambda = \lambda_0$. For the case $\lambda = \lambda_0$ find explicitly the condition on $f(x)$ and, assuming this condition is satisfied, write down the corresponding general solution for $u(x)$. Check that this solution does indeed satisfy the integral equation. (10 points).

b) Use a Laplace transform to find the solution to the generalized Abel equation

$$f(x) = \int_0^x (x-t)^{-\mu} u(t) dt, \quad 0 < \mu < 1,$$

where $f(x)$ is given and $f(0) = 0$, in the form

$$u(x) = \int_0^x K(x-t) f'(t) dt.$$

You should give an explicit expression for the kernel $K(x-t)$. (10 points)

$$(\textbf{Useful Formula:} \quad \int_0^\infty t^{\mu-1} e^{-pt} dt = p^{-\mu}\,\Gamma(\mu), \quad \mu > 0.)$$

— *End* —