

Essential Calculus

We will make heavy use of the following interactions between derivatives and integrals:

a) Integrating a derivative

$$\int_a^b \frac{d}{dx} f(x) dx = f(b) - f(a).$$

b) Differentiating an integral with respect to the upper limit:

$$\frac{d}{db} \int_a^b f(y) dy = f(b).$$

c) Differentiating under the integral sign

$$\frac{d}{dx} \int_a^b f(y, x) dy = \int_a^b \frac{\partial}{\partial x} f(y, x) dy.$$

d) Also useful is Leibnitz' generalization of the last two equations:

$$\frac{d}{dt} \int_{a(t)}^{b(t)} f(x, t) dx = \frac{db}{dt} f(b) - \frac{da}{dt} f(a) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x, t) dx.$$

We can combine (a) with Leibnitz' rule for derivatives

$$\frac{d}{dx} \left(f(x)g(x) \right) = \frac{df(x)}{dx} g(x) + f(x) \frac{dg(x)}{dx}$$

to get the integration-by-parts identity

$$f(b)g(b) - f(a)g(a) = \int_a^b \frac{df(x)}{dx} g(x) dx + \int_a^b f(x) \frac{dg(x)}{dx} dy.$$

This is often written as

$$[fg]_a^b - \int_a^b f'g dx = \int_a^b fg' dx.$$

Integration by parts is one of the most useful tools in calculus.