Q1 Contour Integration: Use the calculus of residues to evaluate the following integrals:

$$\begin{aligned} I_1 &= \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}, \qquad 0<b<a. \\ I_2 &= \int_0^{2\pi} \frac{\cos^2 3\theta}{1-2a\cos 2\theta + a^2} d\theta, \qquad 0<a<1. \\ I_3 &= \int_0^{\infty} \frac{x^\alpha}{(1+x^2)^2} dx, \qquad -1<\alpha<2. \end{aligned}$$

(These are not meant to be easy! You will have to dig for the residues.)

Q2 Lattice Matsubara sums: Show that, for suitable functions $f(z)$, the sum

$$S = \frac{1}{N} \sum_{\omega^N+1=0} f(\omega)$$

of the values of $f(z)$ at the N-th roots of (-1) can be written as an integral

$$S = \frac{1}{2\pi i} \int_C \frac{dz}{z} \frac{z^N}{z^N+1} f(z).$$

Here C consists of a pair of oppositely oriented concentric circles. The annulus formed by the circles should include all the roots of unity, but exclude all singularites of f. Use this result to show that, for N even,

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{\sinh E}{\sinh^2 E + \sin^2 \frac{(2n+1)\pi}{N}} = \frac{1}{\cosh E} \tanh \frac{NE}{2}. \quad (\star)$$

Take the $N \to \infty$ limit while scaling $E \to 0$ in some suitable manner, and hence show that

$$\sum_{n=-\infty}^{\infty} \frac{a}{a^2 + [(2n+1)\pi]^2} = \frac{1}{2} \tanh \frac{a}{2}. \quad (\star\star)$$

Take care not to get this last result wrong by a factor of two: it is *not* true that the limit of the finite sum $(\star)$ is the infinite sum $(\star\star)$.

Q3 Plemelj and Neumann: The Legendre function of the second kind $Q_n(z)$ may be defined for positive integer n by the integral

$$Q_n(z) = \frac{1}{2} \int_{-1}^{1} \frac{(1-t^2)^n}{2^n (z-t)^{n+1}} \, dt, \quad z \notin [-1, 1].$$

Show that for $x \in [-1,1]$ we have

$$Q_n(x+i\epsilon) - Q_n(x-i\epsilon) = -i\pi P_n(x),$$

where $P_n(x)$ is the Legendre Polynomial. Deduce *Neumann 's formula*

$$Q_n(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_n(t)}{z-t}\,dt, \quad z \notin [-1,1].$$

Q4 Hilbert transforms: Suppose that $\varphi_1(x)$ and $\varphi_2(x)$ are real functions with finite $L^2(\mathbb{R})$ norms.

a) Use the Fourier transform result

$$\widetilde{(\mathcal{H}f)}(\omega) = i\,\mathrm{sgn}(\omega)\widetilde{f}(\omega).$$

to show that

$$\langle \varphi_1 | \varphi_2 \rangle = \langle \mathcal{H}\varphi_1 | \mathcal{H}\varphi_2 \rangle.$$

Thus, $\mathcal{H}$ is a unitary transformation from $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

b) Use the fact that $\mathcal{H}^2 = -I$ to deduce that

$$\langle \mathcal{H}\varphi_1 | \varphi_2 \rangle = -\langle \varphi_1 | \mathcal{H}\varphi_2 \rangle$$

and so $\mathcal{H}^\dagger = -\mathcal{H}$.

c) Conclude from part b) that

$$\int_{-\infty}^{\infty} \varphi_1(x) \left(P\int_{-\infty}^{\infty} \frac{\varphi_2(y)}{x-y}\,dy \right) dx = \int_{-\infty}^{\infty} \varphi_2(y) \left(P\int_{-\infty}^{\infty} \frac{\varphi_1(x)}{x-y}\,dx \right) dy,$$

i.e., for $L^2(\mathbb{R})$, functions, it is legitimate to interchange the order of "P" integration with ordinary integration.

d) By replacing $\varphi_1(x)$ by a constant, and $\varphi_2(x)$ by the Hilbert transform of a function f with $\int f\,dx \neq 0$, show that it is not *always* safe to interchange the order of "P" integration with ordinary integration

Q5 Advanced Hilbert transforms:

Suppose that are given real functions $u_1(x)$ and $u_2(x)$ and substitute their Hilbert transforms $v_1 = \mathcal{H}u_1$, $v_2 = \mathcal{H}u_2$ into (9.78) to construct analytic functions $f_1(z)$ and $f_2(z)$. Then the product $f_1(z)f_2(z) = F(z)$ has boundary value

$$F_R(x) + iF_I(x) = (u_1u_2 - v_1v_2) + i(u_1v_2 + u_2v_1).$$

a) By assuming that $F(z)$ satisfies the conditions for (9.77) to be applicable to this boundary value, deduce that

$$\mathcal{H}\left((\mathcal{H}u_1)u_2\right) + \mathcal{H}((\mathcal{H}u_2)u_1) - (\mathcal{H}u_1)(\mathcal{H}u_2) = -u_1u_2. \quad (\star)$$

This result[1] of part (a) sometimes appears in the physics literature[2] in the guise of the distributional identity

$$\frac{P}{x-y}\frac{P}{y-z}+\frac{P}{y-z}\frac{P}{z-x}+\frac{P}{z-x}\frac{P}{x-y}=-\pi^2\delta(x-y)\delta(x-z),$$

where $P/(x-y)$ denotes the principal-part distribution $P\Big(1/(x-y)\Big)$. This attractively symmetric form conceals the fact that x is being kept fixed, while y and z are being integrated over in a specific order. As the next part shows, were we to freely re-arrange the integration order we could use the identity

$$\frac{1}{x-y}\frac{1}{y-z}+\frac{1}{y-z}\frac{1}{z-x}+\frac{1}{z-x}\frac{1}{x-y}=0 \quad x,y,z \text{ distinct}$$

to wrongly conclude that the right-hand side is zero.

b) Show that the identity $(\star)$ can be written as

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{\varphi_1(y)\varphi_2(z)}{(z-y)(y-x)}dz\right)dy=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{\varphi_1(y)\varphi_2(z)}{(z-y)(y-x)}dy\right)dz-\pi^2\varphi_1(x)\varphi_2(x),$$

principal-part integrals being understood where necessary. This is a special case of a more general change-of-integration-order formula

$$\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{f(x,y,z)}{(z-y)(y-x)}dz\right)dy=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\frac{f(x,y,z)}{(z-y)(y-x)}dy\right)dz-\pi^2 f(x,x,x),$$

which is due to G. H. Hardy (1908). It is usually called the *Poincaré-Bertrand theorem*.

[1] F. G. Tricomi, *Quart. J. Math. (Oxford)*, (2) **2**, (1951) 199.
[2] For example, in R. Jackiw, A. Strominger, *Phys. Lett.* **99B** (1981) 133.