Q1 Binomial Series: Show that the binomial series expansion of $(1+x)^{-\nu}$ can be written as

$$(1+x)^{-\nu} = \sum_{m=0}^{\infty}(-x)^m \frac{\Gamma(m+\nu)}{\Gamma(\nu)\, m!}, \quad |x|<1.$$

Q2 A Mellin transform and its inverse: Combine the Beta-function identity with a suitable change of variables to evaluate the Mellin transform

$$\int_0^\infty x^{s-1}(1+x)^{-\nu}\, dx, \quad \nu>0,$$

of $(1+x)^{-\nu}$ as a product of Gamma functions. Now consider the *Bromwich contour* integral

$$\frac{1}{2\pi i\, \Gamma(\nu)}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(\nu-s)\Gamma(s)\, ds.$$

Here $\mathrm{Re}\, c \in (0,\nu)$. The contour therefore runs parallel to the imaginary axis with the poles of $\Gamma(s)$ to its left and the poles of $\Gamma(\nu-s)$ to its right. Use the identity

$$\Gamma(s)\Gamma(1-s) = \pi\, \mathrm{cosec}\, \pi s$$

to show that when $|x|<1$ the contour can be closed by a large semicircle lying to the left of the imaginary axis. By using the preceding exercise to sum the contributions from the enclosed poles at $s=-n$, evaluate the integral and so verify that the Bromwich contour provides the inverse of the Mellin transform in this case.

Q3 Mellin-Barnes integral.: Use the technique developed in the preceding exercise to show that

$$F(a,b,c;-x) = \frac{\Gamma(c)}{2\pi i\, \Gamma(a)\Gamma(b)}\int_{c-i\infty}^{c+i\infty} x^{-s}\frac{\Gamma(a-s)\Gamma(b-s)\Gamma(s)}{\Gamma(c-s)}\, ds,$$

for a suitable range of x. This integral representation of the hypergeometric function is due to the English mathematician Ernest Barnes (1908), later a controversial Bishop of Birmingham.

Q4 Conformal block equation: Let

$$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Show that the matrix differential equation

$$\frac{d}{dx}Y = \frac{A}{z}Y + \frac{B}{1-z}Y,$$

where

$$A = \begin{pmatrix} 0 & a \\ 0 & 1-c \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ b & a+b-c+1 \end{pmatrix},$$

has a solution

$$Y(z) = F(a,b,;c,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{z}{a}F'(a,b;c;z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(This result is useful in conformal field theory)