1) Infinitesimal Homotopy: Use the infinitesimal homotopy relation to show that the Lie derivative $\mathcal{L}$ commutes with the exterior derivative d, *i.e.* for ω a p-form, we have

$$d\left(\mathcal{L}_X\omega\right) = \mathcal{L}_X(d\omega).$$

2) Magnetic solid: The semi-classical dynamics of charge $-e$ electrons in a magnetic solid are governed by the equations

$$\begin{aligned}\dot{\mathbf{r}} &= \frac{\partial\epsilon(\mathbf{k})}{\partial\mathbf{k}} - \dot{\mathbf{k}}\times\mathbf{\Omega},\\ \dot{\mathbf{k}} &= -\frac{\partial V}{\partial\mathbf{r}} - e\dot{\mathbf{r}}\times\mathbf{B}.\end{aligned}$$

Here $\mathbf{k}$ is the Bloch momentum of the electron, $\mathbf{r}$ is its position, $\epsilon(\mathbf{k})$ its band energy (in the extended-zone scheme), and $\mathbf{B}(\mathbf{r})$ is the external magnetic field. The components Ω_i of the *Berry curvature* $\mathbf{\Omega}(\mathbf{k})$ are given in terms of the periodic part $|u(\mathbf{k})\rangle$ of the Bloch wavefunctions of the band by

$$\Omega_i(\mathbf{k}) = i\epsilon_{ijk}\frac{1}{2}\left(\left\langle\frac{\partial u}{\partial k_j}\middle|\frac{\partial u}{\partial k_k}\right\rangle - \left\langle\frac{\partial u}{\partial k_k}\middle|\frac{\partial u}{\partial k_j}\right\rangle\right).$$

The only property of $\mathbf{\Omega}$ needed for the present problem, however, is that $\mathrm{div}_{\mathbf{k}}\mathbf{\Omega} = 0$.

a) Show that these equations are Hamiltonian, with

$$H(\mathbf{r},\mathbf{k}) = \epsilon(\mathbf{k}) + V(\mathbf{r})$$

and

$$\omega = dk_i dx_i - \frac{e}{2}\epsilon_{ijk}B_i(\mathbf{r})dx_j dx_k + \frac{1}{2}\epsilon_{ijk}\Omega_i(\mathbf{k})dk_j dk_k.$$

as the symplectic form.

b) Confirm that the ω defined in part b) is closed, and that the Poisson brackets are given by

$$\begin{aligned}\{x_i, x_j\} &= \frac{\epsilon_{ijk}\Omega_k}{(1+e\mathbf{B}\cdot\mathbf{\Omega})},\\ \{x_i, k_j\} &= -\frac{\delta_{ij}+e\Omega_i B_j}{(1+e\mathbf{B}\cdot\mathbf{\Omega})},\\ \{k_i, k_j\} &= +\frac{\epsilon_{ijk}eB_k}{(1+e\mathbf{B}\cdot\mathbf{\Omega})}.\end{aligned}$$

c) Show that the conserved phase-space volume $\omega^3/3!$ is equal to

$$(1 + e\mathbf{B} \cdot \mathbf{\Omega}) d^3k d^3x,$$

instead of the textbook $d^3k d^3x$.

3) Non-abelian gauge fields as matrix-valued forms: In a non-abelian gauge theory, such as QCD, the vector potential

$$A = A_\mu dx^\mu$$

becomes matrix-valued, meaning that the components, A_μ, are matrices that do not necessarily commute with each other. The matrix-valued field-strength F is a 2-form defined by

$$F = dA + A^2 = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu.$$

Here, a combined matrix and wedge product is to be understood:

$$(A^2)_{ik} \equiv \sum_j A_{ij} \wedge A_{jk} = \sum_j A_{ij;\mu} A_{jk;\nu} \, dx^\mu dx^\nu.$$

i) Show that $A^2 = \frac{1}{2}[A_\mu, A_\nu] dx^\mu dx^\nu$, and hence show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

ii) Define *gauge-covariant derivatives*

$$\nabla_\mu = \partial_\mu + A_\mu,$$

and show that the commutator of two of these is equal to

$$[\nabla_\mu, \nabla_\nu] = F_{\mu\nu}.$$

iii) Let g be an invertable matrix, and δg a matrix describing a small change in g. Show that the corresponding change in the inverse matrix is given by $\delta(g^{-1}) = -g^{-1}(\delta g) g^{-1}$.

iv) Show that a necessary condition for the matrix-valued gauge field A to be "pure gauge", *i.e.* for there to be a position dependent matrix g such that $A = g^{-1} dg$, is that $F = 0$.

v) Show that under the *gauge transformation*

$$A \to A^g \equiv g^{-1} A g + g^{-1} dg,$$

we have $F \to g^{-1} F g$. (Hint: The labour is minimized by exploiting the covariant derivative identity in ii)).

vi) Show that F obeys the *Bianchi identity*

$$dF - FA + AF = 0.$$

This equation is the non-abelian version of the source-free Maxwell equations.

vii) Show that, in any number of dimensions, the Bianchi identity implies that the 4-form $\mathrm{tr}\,(F^2)$ is closed, *i.e.* that $d\,\mathrm{tr}\,(F^2) = 0$. (The trace is being taken only over the matrix indices.)

viii) Show that,

$$\mathrm{tr}\,(F^2) = d\left\{\mathrm{tr}\,(AdA + \frac{2}{3}A^3)\right\},$$

so that if $\int_\Omega \mathrm{tr}\,(F^2) \neq 0$, and $\partial\Omega = \emptyset$, then there cannot be a globally-defined A on the region Ω. The 3-form $\mathrm{tr}\,(AdA + \frac{2}{3}A^3)$ is called a *Chern-Simons* form.

When the gauge group is $\mathrm{SU}(n)$, the integral

$$c_2(A) = \frac{1}{8\pi^2}\int_{\mathbf{R}^4} \mathrm{tr}\,(F^2)$$

is an integer-valued topological invariant called the *Chern number*, or *instanton number*, of the gauge field configuration A.

The $2n$-forms $\mathrm{tr}\,(F^n)$ are also closed, and can locally be written as the d of $(2n-1)$-form generalizations of the Chern-Simons form.