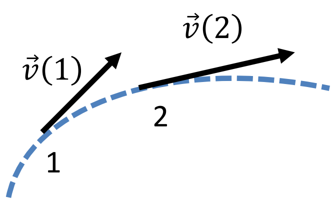
Acoustics

Intro to Fluid Mechanics

Let's begin with some simple fluid mechanics. By fluids I mean liquids and gases – media that cannot maintain a shear force. In continuum mechanics we deal with tiny packets containing a huge number of molecules but small on the scale of the spatial disturbances we're interested in. The fluid will be described by its density ρ, pressure p and the velocity $\vec{v}$ of a packet of fluid at the point $\vec{r}$ and time t. First, we have conservation of mass, described in the same way we would conservation of charge, by an *equation of continuity*.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

In other words, the density at any point will change according to the divergence of mass flux $\rho\vec{v}$. To describe how $\vec{v}$ changes with time, we need Newton's second law, *F* = *ma*. Remember that $\vec{v}$ is the fluid velocity defined at each point in space, just like temperature or pressure. But Newton's law refers to an *individual* packet of fluid as it moves through space.



Suppose the packet moves from point 1 to point 2 along the dotted path. The acceleration is given by change in its velocity for small $\Delta x, \Delta y, \Delta z, \Delta t$. For example, the acceleration in the x-direction is given by,

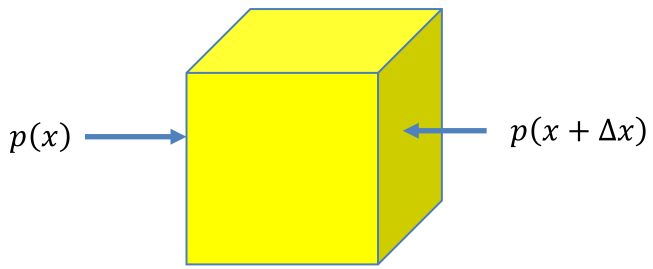
$$a_x = \frac{\Delta v_x}{\Delta t} = \frac{v_x(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - v_x(x,y,z,t)}{\Delta t}$$

$$\approx \frac{\partial v_x}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial v_x}{\partial y}\frac{\Delta y}{\Delta t} + \frac{\partial v_x}{\partial z}\frac{\Delta z}{\Delta t} + \frac{\partial v_x}{\partial t} \to (\vec{v}\cdot\nabla)v_x + \frac{\partial v_x}{\partial t}$$

This expression with the gradient along with the explicit time derivative is called the *convective* derivative. A similar expression holds for each velocity component. Take a packet of volume ΔV. This has a mass $\rho\Delta V$ so Newton's law for the packet is given by,

$$\rho \Delta V \left[(\vec{v}\cdot\nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} \right] = \vec{F}$$

where $\vec{F}$ is the force on the packet. The force can come from from pressure, gravity and viscosity. We're interested in sound waves for which pressure is the most important force.



Pressure is force per unit area. It acts normal to any surface. Consider an infinitesimal cube of fluid in which the pressure varies from place to place. Concentrate on the two surfaces perpendicular to the x-axis, located at x and $\mathrm{x}+\Delta x$. The net force in the x direction will be the net pressure times the cross-sectional area:

$$F_x = \big(p(x) - p(x+\Delta x)\big)\,\Delta y \Delta z = -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z = -\frac{\partial p}{\partial x}\Delta V$$

We can do the same calculation for the other 4 faces. This leads to derivatives of the pressure in the y and z directions. We arrive at the vector force on the volume,

$$\vec{F} = -\nabla p\, \Delta V$$

Inserting this into our previous expression of Newton's law,

$$\rho \Delta V\left[(\vec{v}\cdot\nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t}\right] = \vec{F} = -\nabla p\,\Delta V$$

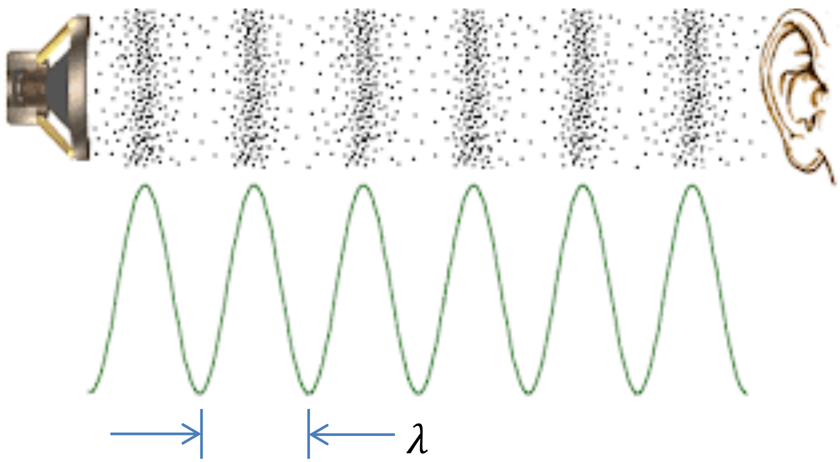
Dividing both sides by $\rho\Delta V$ we obtain *Euler's equation* for a non-viscous fluid,

$$(\vec{v}\cdot\nabla)\vec{v} + \frac{\partial \vec{v}}{\partial t} = -\frac{\nabla p}{\rho}$$

For much of fluid mechanics – water flowing through pipes, for example, it is a very good approximation to assume the the fluid is *incompressible*, meaning that the density is a constant. However, for sound waves that's not true. The density definitely *does* change so we'll be in the regime of *compressible* fluid flow. However, the good news is that except for very large amplitude sound waves, we can ignore the nasty $(\vec{v}\cdot\nabla)\vec{v}$ term. What about viscosity? That will add a complicated term on the right hand side to Euler's equation. And viscosity is vital to describe much of anything interesting for incompressible fluid flow – airplane wings, water flowing down pipes, turbulence, for example. For sound, we can understand a great deal by first ignoring viscosity. Its main effect will be to sap energy out of a sound wave.

Sound Waves

Sound is a disturbance that propagates through gases, liquids, solids and even a plasmas. The figure below illustrates the situation for a sound a wave in air. Imagine a loudspeaker subjected to a sinusoidal voltage at some frequency, say 1000 Hz. The electrical signal moves the speaker membrane back and forth at 1000 Hz, which in turn pushes on the air adjacent to it. This motion of the speaker membrane creates regions of high and low pressure that propagate out from the speaker at speed c_S. An imaginary snapshot of the gas molecules (greatly exaggerated) would reveal a periodic arrangement of high and low density regions. These successive wavefronts move a delicate membrane inside your ear whose vibrations eventually move tiny "cilia" back and forth, generating nerve impulses that are interpreted as sound by your brain.



https://physicsofscifi.blogspot.com/2012/07/star-trek-sound-in-outer-space-and-on.html

Sound waves correspond to *compressible* fluid flow, as opposed to the flow of liquids through pipes and down rivers in which the fluid may be considered incompressible. As with electromagnetic waves, the wavelength and frequency are simply related to the sound wave velocity,

$$f\,\lambda = c_S$$

where f is the frequency in Hertz and λ is the wavelength in meters. Successive peaks in the air pressure are separated by λ. The sound velocity is a property of the medium. For air at $T = 25°\mathrm{C}$, $c_S = 764\,mph \approx 300\,m/sec$. Our treatment is applicable to gases and liquids and follows the text *Fluid Mechanics* by Landau and Lifshitz.

Call the ambient pressure p_0 and the ambient density ρ_0 both of which we'll assume are constant everywhere. A sound wave is a disturbance in both quantities that propagates. Call these disturbances p' and ρ'. For most sound waves of interest $p' \ll p_0$ and $\rho' \ll \rho_0$. There is also an associated *fluid velocity* $\vec{v}$. Think of a water wave at the beach. If you stand in place the water move up and down with a local velocity $\vec{v}$ that depends on where you're standing at a given time. $\vec{v}$ is not the speed of the wave, nor is it the velocity of any individual water molecule. It's the local velocity of a small packet of water. Similarly with a sound wave, there will be a fluid velocity $\vec{v}$ that accompanies the disturbances in pressure and density. Letting ρ and p be the total density and pressure of the fluid, the necessary equations are continuity and Euler's equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad\qquad \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p$$

Sound waves are small disturbances so we can make the approximations,

$$p = p_0 + p' \quad p' \ll p_0 \qquad \rho = \rho_0 + \rho' \quad \rho' \ll \rho_0$$

Assume (to be shown later) that $\vec{v}$ is also small compared to *both* the wave velocity and the molecular velocity. Then terms involving products of $\vec{v}, \rho'$ or p' can be considered higher order and ignored. The continuity equation and Euler equations can then be linearized:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \vec{v} = 0 \qquad \frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0 \qquad p' \ll p_0 \quad \rho' \ll \rho_0$$

We now have 2 equations with 3 unknowns ($\vec{v}, \rho', p'$) so another equation is needed; a relationship between the the p and ρ. For an ideal gas you might be tempted to use the ideal gas law $p = \rho k_B T/m$ where *m* is the molecular mass. However, that generally gives much too small a sound velocity. The disturbances in sound waves usually oscillate too rapidly for the fluid to remain at a uniform temperature. A much better approximation is to assume that pressure and density disturbances occur with no flow of heat or *adiabatically*, meaning that a given fluid packet expands and contracts at constant *entropy* S. The pressure disturbance in the sound wave will therefore be written as,

$$p = p_0 + p' \approx p_0 + \left(\frac{\partial p}{\partial \rho_0}\right)_S \rho' \quad \rightarrow \quad p' = \left(\frac{\partial p}{\partial \rho_0}\right)_S \rho'$$

The quantity $\left(\frac{\partial p}{\partial \rho_0}\right)_S$ is a property of the fluid that will be determined momentarily from thermodynamics. Using the 3 equations we can now move on to find a wave equation. Specialize to the case of a wave moving in the x-direction. Sound waves in fluids are *longitudinal*, meaning that the fluid velocity is along the direction of wave propagation. (Both longitudinal and transverse waves may propagate in solids but that will come later.) Assume motion along the x direction so $\vec{v} = v_x(x,t)\,\hat{x}$. Eliminating p' leaves two equations,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_x}{\partial x} = 0 \qquad \frac{\partial v_x}{\partial t} + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho_0}\right)_S \frac{\partial \rho'}{\partial x} = 0$$

Taking the time derivative of the first equation and the space derivative of the second and setting $\frac{\partial^2 v_x}{\partial t \partial x} = \frac{\partial^2 v_x}{\partial x \partial t}$ leaves a wave equation,

$$\frac{\partial^2 \rho'}{\partial t^2} = \left(\frac{\partial p}{\partial \rho_0}\right)_S \frac{\partial^2 \rho'}{\partial x^2} = c_S^2 \frac{\partial^2 \rho'}{\partial x^2} \qquad c_S = \sqrt{\left(\frac{\partial p}{\partial \rho_0}\right)_S}$$

A sound wave therefore travels with a phase velocity,

$$c_s = \sqrt{\left(\frac{\partial p}{\partial \rho_0}\right)_S}$$

It's easy to see that all three quantities, density, pressure and fluid velocity, obey the same wave equation.

Speed of sound in an ideal gas

Consider an ideal gas whose equation of state is given by $p_0 = nk_BT = \rho_0 k_B T/m$ where *n* is the density in molecules per cubic meter and *m* is the mass of one molecule. Thermodynamics tells us that when an ideal gas undergoes a a change at constant entropy the following equation holds,

$$p_0 V^\gamma = constant \qquad \gamma = \frac{C_P}{C_V}$$

where C_P and C_V are the heat capacities measured at constant pressure and constant volume, respectively. For a simple monatomic gas like helium $\gamma = 5/3$. For diatomic gases like N_2 or O_2, $\gamma = 7/5$ at room temperature. Taking derivatives we have,

$$0 = d(p_0 V^\gamma) = dp\, V^\gamma + p_0 \gamma\, V^{\gamma-1}\, dV \;\; \rightarrow \;\; dp = -\,p_0 \gamma \frac{dV}{V}$$

We need the change in pressure with respect to density, not volume. To get that, consider a volume *V* containing *N* gas molecules. Its density is given by $\rho_0 = mn = mN/V$. Changing the volume by a small amount,

$$\rho_0 + d\rho_0 = \frac{mN}{V + dV} \approx \rho_0 - \rho_o \frac{dV}{V} \;\; \rightarrow \;\; d\rho_0 = -\rho_o \frac{dV}{V}$$

Eliminating dV/V from these two equations and using the ideal gas law, $p_0 = \rho_0 k_B T/m$ we obtain,

$$\left(\frac{\partial p}{\partial \rho_0}\right)_S = \frac{\gamma k_B T}{m} = c_S^2$$

For sound waves in air (78% N_2 and 21% O_2 so $\gamma = 7/5$) c_S = 340 m/sec at *T* = 293 Kelvin.

It's interesting to compare the sound velocity c_S to the average molecular speed V in the gas. This quantity can be estimated from the equipartition theorem of statistical mechanics, which says that the statistical average of any quadratic term in the energy is given by $\langle E \rangle = k_B T/2$. Applying this theorem to the kinetic energy of a molecule, there are 3 quadratic terms,

$$\langle Kinetic\ energy \rangle = \left\langle \frac{1}{2} m\left(V_x^2 + V_y^2 + V_z^2\right) \right\rangle = 3 * \frac{k_B T}{2} \quad \rightarrow \langle V^2 \rangle = \frac{3 k_B T}{m} \quad \leftrightarrow \quad \frac{\gamma k_B T}{m} = c_s^2$$

Comparing $\langle V^2 \rangle$ to c_S^2 , we see that they are comparable since $\gamma = 5/3$ is between 1 and 2. Next, let's see how the the magnitude of the fluid velocity $\vec{v}$ compares to c_S.

Decibels

We're all used to hearing about loudness in terms of *decibels*. Remember that the pressure disturbance *p*' also obeys the wave equation so consider at a simple travelling wave solution,

$$p' = p'_{peak} \cos(kx - \omega t)$$

where $\omega/k = c_S$ is the phase velocity of the wave. The root mean-square of the pressure amplitude is defined as $p'_{RMS} = p'_{peak}/\sqrt{2}$. The sound pressure in decibels (measured in dBSPL) is defined as,

$$p'(dBSPL) = 20\ log_{10} \frac{p'_{RMS}(Pascals)}{2\ x 10^{-5}}$$

Atmospheric pressure at sea level is about 10^5 Pascals. Sound waves, even very loud ones, involve far smaller pressures disturbances. A 120 dBSPL sound wave (right at the pain threshold for humans) has an RMS pressure of $p'_{RMS} = 10^6\ x\ 2\ x\ 10^{-5} = 20\ Pascals$ which is only $20/10^5$ = 0.0002 times atmospheric pressure. This justifies our approximation that $p' << p_0$ for any sound that we can bear to hear.

The next table shows pressure levels in decibels for various sounds. The human ear can detect sounds over more than 14 orders of magnitude in intensity so a logarithmic scale appropriate. On the other hand, shock waves that might come from a nearby explosion can have pressure amplitudes comparable to atmospheric pressure, close to 200 dB, which is far beyond the pain threshold.

Sound waves can also be characterized by their intensity in Watts/m^2.

$$I(dB) = 10\ log_{10} \frac{I\ (W/m^2)}{10^{-12}}$$

On this scale and intensity of 10^{-12} W/m^2 (0 dB) is at to the threshold of hearing for most people. It corresponds to the eardrum moving less than the diameter of an atom.

Examples	Sound Pressure Level L_p dBSPL
Jet aircraft, 50 m away	140
Threshold of pain	130
Threshold of discomfort	120
Chainsaw, 1m distance	110
Disco, 1 m from speaker	100
Diesel truck, 10 m away	90
Kerbside of busy road, 5 m	80
Vacuum cleaner, distance 1 m	70
Conversational speech, 1m	60
Average home	50
Quiet library	40
Quiet bedroom at night	30
Background in TV studio	20
Rustling leaf	10
Threshold of hearing	0

https://www.omnicalculator.com/physics/db

Assuming $p' << p_0$ we can go now back and estimate the size of the fluid velocity v_x . Recall the linearized version of Euler's equation in one dimension,

$$\frac{\partial v_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = 0$$

Again, assume sinusoidal solutions for the pressure and velocity,

$$p' = p'_{peak}\, cos(kx - \omega t) \qquad v_x = v_{x_{peak}}\, cos(kx - \omega t)$$

Plugging these into Euler's equation, using $\omega/k = c_S$ and the ideal gas law, $p_0 = nk_BT$ (valid for the equilibrium density) we have,

$$v_{x_{peak}} = \frac{p'_{peak}}{\rho_0\, c_s} = \frac{p'_{peak}}{\rho_0\, c_s^2} c_s = \frac{p'_{peak}}{mn\dfrac{\gamma k_B T}{m}} c_s = \frac{p'_{peak}}{\gamma p_0} c_s \qquad \rightarrow \qquad \frac{v_{peak}}{c_s} = \frac{p'_{peak}}{\gamma p_0} \ll 1$$

The inequality holds because γ is of order 1-2 and we've just shown than the pressure disturbance p'_{peak} in a sound wave is tiny compared to the ambient pressure p_0 . The net result is that $v_x \ll c_s$. That's *not* true for shock waves but shock waves involve nonlinear solutions to the equations of fluid mechanics and are outside the scope of what we just derived. Ordinary sound waves occur in the linear approximation.

Acoustic impedance

The previous equation also brought out an important relationship between the fluid velocity and pressure amplitude in a sound wave,

$$p'_{peak} = (\rho_0\, c_s) v_{x_{peak}} = Z_s\, v_{x_{peak}}$$

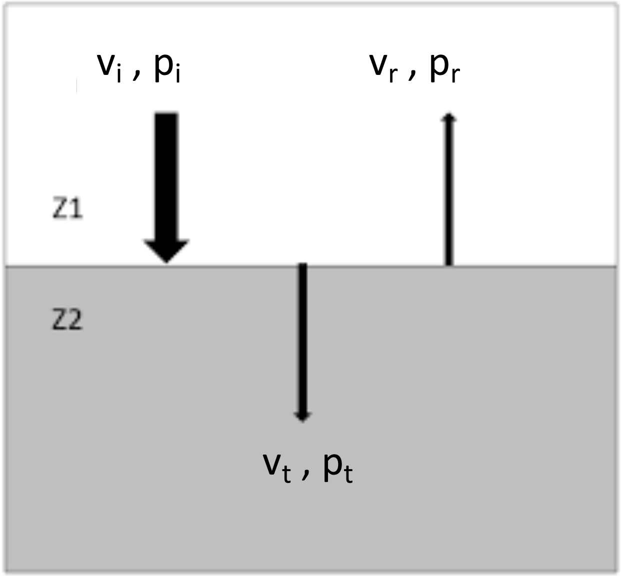
$Z_s = \rho_0\, c_s$ is called the *acoustic impedance* of the medium. In SI units, it has units of Pascal-sec/meter = Rayl . Z_s is particularly useful when examining sound waves as they travel from one medium into another, i.e., how much is reflected and how much is transmitted. I've included a table showing the parameters for a variety of different materials. Many of these are metals and plastics, both of which support longitudinal sound waves, although they do not obey the equations of fluid dynamics.

No.	Materials	Density (kg/m^3)	Sound speed (m/s)	Impedance (Rayl)	Ref.
1	air	1.23	340	418	[7]
2	Water	1053	1490	1.56×10^6	[28]
3	Cork	0.24	530	1.27×10^5	[29]
4	Aluminum	2690	6420	1.73×10^7	[30]
5	Steel	7860	5950	4.64×10^7	[30]
6	PDMS	969	1119	1.08×10^6	[28]
7	PU	1528	1040	1.59×10^6	[31]
8	Epoxy	1180	2490	2.95×10^6	[32]
9	Hydrogel	1000	1600	1.60×10^6	[33]
10	Ecoflex	1070	989	1.06×10^6	[34]

https://www.researchgate.net/figure/Characteristic-specific-acoustic-impedances-of-polymer-materials-in-comparison-to-some_tbl1_349089661

Reflection and transmission

Suppose we have a sound wave normally incident on the boundary between two media with acoustic impedance Z_1 and Z_2 respectively. Let's see how much sound is transmitted and how much is reflected. There is one point before we go on. If you go back to the Euler equation and plug in solutions for *left-going* waves, $p' = p'_{peak}\, cos(kx + \omega t)$ and $v_x = v_{x_{peak}}\, cos(kx + \omega t)$ then the impedance is $Z_s = -\, \rho_0\, c_s$ so there is an important (-) sign. Now, as with electromagnetic waves, apply boundary conditions.



We'll use the notation p_i, p_r and p_t for the pressure amplitudes of the incident, reflected and transmitted sound waves and similarly for the fluid velocity amplitudes. Both the total pressure and total velocity must be continuous at the boundary. Therefore,

$$p_i + p_r = p_t \qquad v_i + v_r = v_t$$

Using acoustic impedances the first equation becomes,

$$Z_1 v_i - Z_1 v_r = Z_2 v_t$$

Solving for the reflection coefficient we have,

$$R \equiv \frac{v_r}{v_i} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

We are usually more interested in the transmission and reflection of the sound *energy*. The full derivation is somewhat lengthy (see Landau and Lifshitz, *Fluid Mechanics*) but the result is easy to remember. For a plane wave, the intensity I = energy flux = acoustic energy passing through a square meter per second, is given by the product of the wave velocity and the energy density, as with electromagnetic waves,

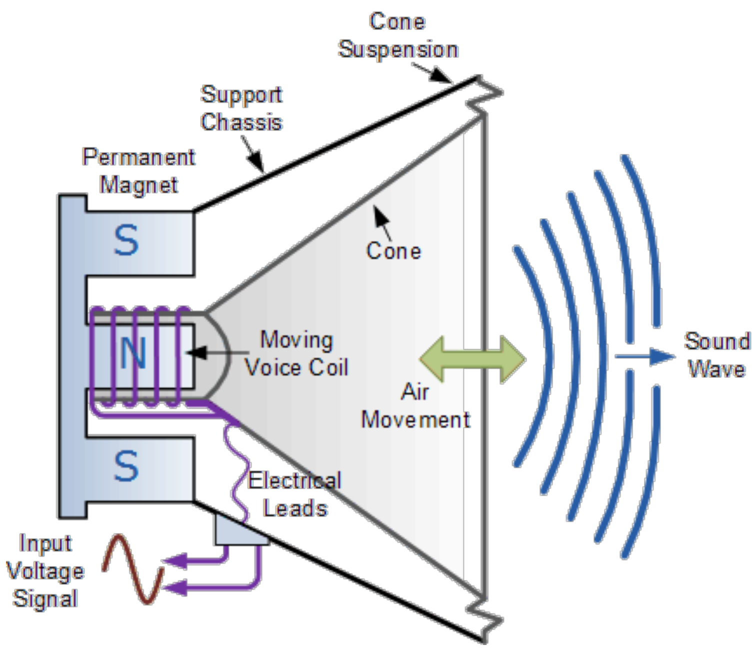
$$I = c_S(Energy\ density) = c_S \rho_0 v^2 = Z v^2$$

The coefficients of reflected and transmitted intensity are obtained by squaring *R* and using the conservation of energy,

$$\frac{I_r}{I_i} = \left|\frac{Z_1 - Z_2}{Z_1 + Z_2}\right|^2 \qquad \frac{I_t}{I_i} = 1 - \frac{I_r}{I_i} = \frac{4\, Z_1 Z_2}{(Z_1 + Z_2)^2}$$

A large mismatch in acoustic impedances implies lots of reflected energy and little energy transmitted. This will be important when we discuss transducers for medical imaging.

Generation of sound waves



https://www.electronics-tutorials.ws/io/io_8.html

Pretty much anything that moves through the air will generate sound. Predicting the precise form of the resulting wave from the specific shape of the vibrating object is very complicated. As such, the design of musical instruments is as much an art as a science and outside the scope of these lectures and of my expertise. As an example, consider a loudspeaker whose essential parts are shown in the figure.

A coil of wire is attached to the back of the cone and inside the coil sits a permanent magnet. A sinusoidal current passing through the coil generates a sinusoidal Lorentz force acting on the coil which moves the cone back and forth. That movement generates oscillations of the density in the nearby air which then propagate away as sound waves. Depending on the size and shape of the speaker it generates sound waves in the audio range of frequencies (10 Hz – 20 kHz).

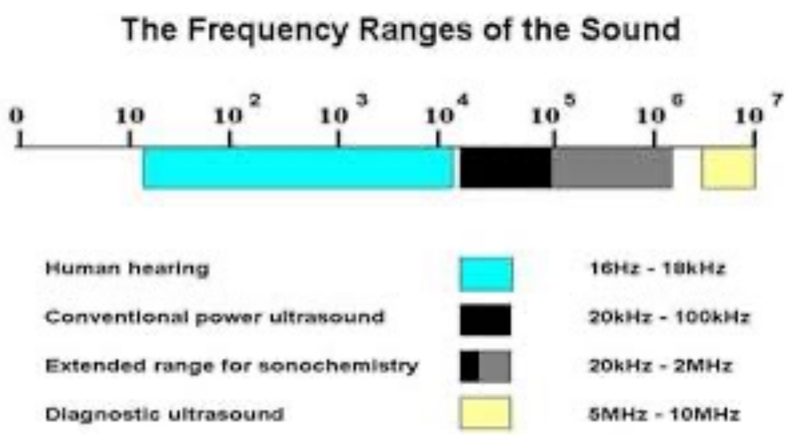
Ultrasound

Sound with frequencies above the human audio range of 20 kHz is termed *ultrasound*. The table shows that several different animals can detect sounds above 100 kHz. Bats, for example, navigate by emitting ultrasonic waves and then detecting the resulting echo from nearby objects. Even though we cannot hear it, ultrasound has many uses. We've all seen ultrasound images of a developing fetus. In addition to medical technology ultrasound is used for cleaning fine parts and silicon wafers, range finding, testing weld joints and and promoting chemical reactions (*sonochemistry*).

Animal	Hearing range in Hertz
Humans	20 – 20,000
Bats	2000 – 110,000
Elephant	16 – 12,000
Fur Seal	800 – 50,000
Beluga Whale	1000 – 123,000
Sea Lion	450 – 50,000
Harp Seal	950 – 65,000
Harbor Porpoise	550 – 105,000
Killer Whale	800 – 13,500
Bottlenose Dolphin	90 – 105,000
Porpoise	75 – 150,000
Dog	67 – 45,000
Cat	45 – 64,000
Rat	200 – 76,000
Opossum	500 – 64,000
Chicken	125 – 2,000
Parakeet	200 – 8,500
Horse	55 – 33,500

https://www.compadre.org/osp/EJSS/4489/274.htm

The lower figure shows the approximate frequency ranges used in various applications of ultrasound. Ultrasonic cleaners operate in the 20 – 200 kHz range. Sonochemistry operates up to a few MHz. Medical imaging is generally done with ultrasound in the 2-15 MHz range.

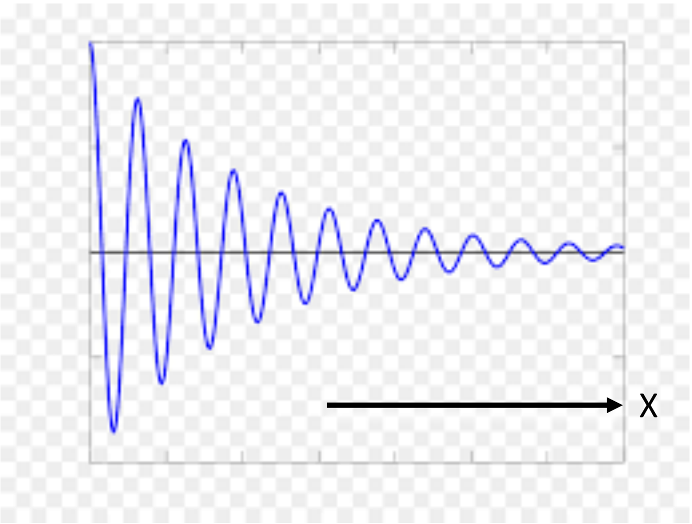


https://www.engineersgarage.com/ultrasonic-uv-sensors-or-ultrasound-sensors/

Attenuation of Sound

You'll notice that the above figure shows an upper limit for medical ultrasound of around 10– 15 MHz. As with light waves, higher resolution can be achieved with shorter wavelengths and therefore higher frequencies. So why not operate at say 400 MHz? The problem is that sound waves loose energy as they travel along and that severely limits the range of operation for something like medical imaging. Sound attenuation is defined by the coefficient α which describes how the intensity of the wave decreases as we move a distance x,

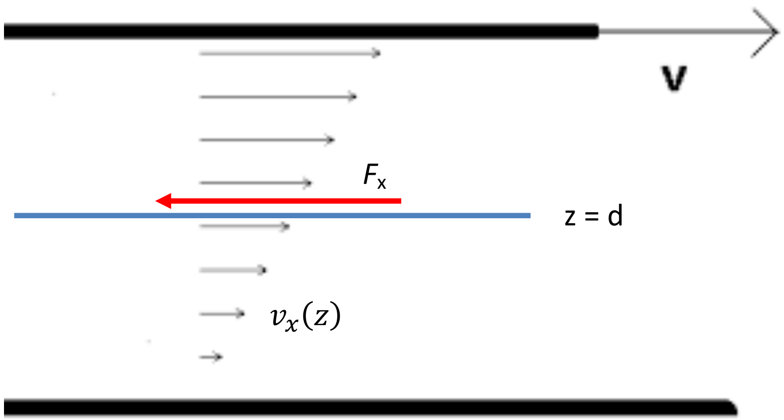
$$I(x) = I(0)\, e^{-\alpha x}$$



It's usually measured in decibels/meter so 10 decibels would correspond to a change of intensity by a factor of 10. As we'll see, the attenuation generally increases rapidly with frequency. To understand this we first need to discuss viscosity.

Viscosity

To remind you about viscosity, imagine we have a fluid constrained between two infinite plates, one at z = 0 and one at z = d. Suppose the top plate moves along at uniform velocity V in the x-direction, while the lower plate is stationary. We'll assume that fluid immediately next to the plate moves at the same speed as the plate. Then the local velocity of the fluid would vary with z as shown. Call it $v_x(z)$. The fluid velocity field has a *shear.*

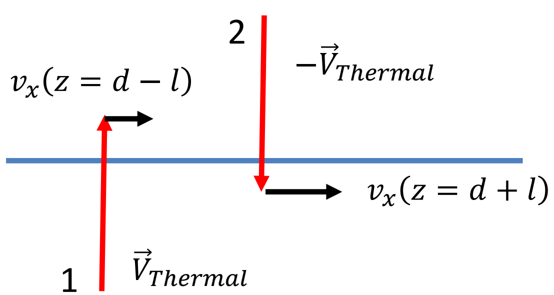


Focus on some plane, say z = d, shown by the blue line. Fluid below this plane exerts a force in the x-direction on fluid above this plane. Consider just a square of area A in this plane. We'll call this force on this area of fluid,

$$F_x = -\eta \frac{\partial v_x}{\partial z} A$$

where η is the viscosity. Since $\partial v_x / \partial z > 0$ and η is defined to be positive, the (-) sign out front is required so the force acts in the negative x direction, as shown by the red arrow. It's a drag or friction force.

To understand where viscosity comes from, refer to the figure below. Recall that for each molecule the total velocity is the sum of a very large thermal velocity and a much smaller velocity v_x coming from the fluid flow. Molecule 1 starts out at z = d-l . It has a big upward thermal velocity and a tiny fluid velocity v_x(d-l). When it crosses the blue plane it delivers some x-momentum mv_x(d-l) to the top fluid. Molecule 2 starts out at d+l with a big downward thermal velocity. It crosses the boundary and delivers a slightly *bigger* x-momentum mv_x(d+l) to the fluid below. The top fluid is therefore *losing* x-momentum. Since force is the change in momentum per second, there is negative-going force F_x on the top fluid.



To get the viscosity coefficient, multiply the number of molecules per unit volume (*n*) by $V_{Thermal}$ and by the *change* in x-momentum of the top fluid for each process like the one shown in the figure. That gives the net x-force per unit area:

$$\frac{F_x}{A} = \frac{1}{6} mnV_{Thernal}\left(v_x(d-l) - v_x(d+l)\right) \approx -\frac{1}{3} nml\, V_{Thermal} \frac{\partial v_x}{\partial z} = -\eta \frac{\partial v_x}{\partial z}$$

The 1/6 is there because on average 1/6 of the molecules have a thermal velocity in the +z direction. *l* is the mean free path: how far, on average, a molecule travels before it collides with another one. Viscosity is proportional to all these quantities. We just found the force on the plate. Viscosity adds a new force, in addition to pressure. When we calculate the viscous force on a little volume of fluid we end up with the full equation for a viscous fluid,

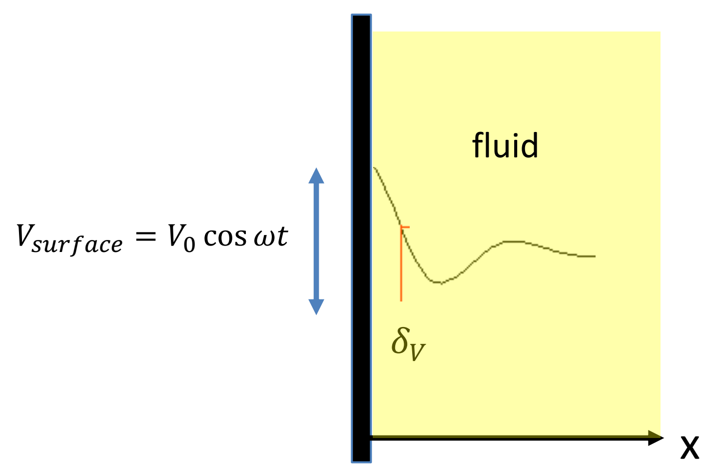
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p}{\rho} + \frac{\eta}{\rho} \nabla^2 \vec{v} + \left(\eta + \frac{1}{3}\zeta\right) \nabla(\nabla \cdot \vec{v})$$

It's impossible to solve exactly except for a few special cases. There is also another coefficient ζ called second viscosity related to internal degrees of freedom in the material. We'll focus on some easy-to-solve cases that are highly relevant to sound waves.

Viscous penetration depth

We know E&M that a high frequency wave can't fully penetrate a metal. The electric field inside the conductor drops exponentially,

$$|E(x)| = |E(x=0)|e^{-x/\delta} \qquad \delta = \sqrt{\frac{\rho}{\pi\mu f}}$$



Something similar happens in viscous fluids and it illustrates why you can't propagate a transverse wave in a fluid. Suppose we take a plate with a fluid adjacent to it and oscillate the plate up and down in the y-direction at frequency ω. Fluids generally obey a no-slip condition at surfaces, meaning that the fluid velocity must match the surface velocity. So at the surface, defined by x = 0, we have,

$$v_y(x=0,t) = V_{surface} = V_0 \cos \omega t$$

The velocity obeys the fluid equation which thankfully simplifies to,

$$\frac{\partial v_y}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = \frac{\eta}{\rho_0}\frac{\partial^2 v_y}{\partial x^2}$$

Since the motion is transverse to the direction of propagation, the continuity equation becomes,

$$0 = \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) \approx \frac{\partial \rho'}{\partial t} + \rho_0\frac{\partial v_y}{\partial y} = \frac{\partial \rho'}{\partial t} + 0 \quad \rightarrow \quad \rho' = constant = 0$$

The density doesn't change with time and since the density variations ρ' are proportional to the pressure variations p' , then $p' = 0$ also. The equation of motion reduces to,

$$\frac{\partial v_y}{\partial t} = \frac{\eta}{\rho_0}\frac{\partial^2 v_y}{\partial x^2}$$

Assume a transverse wave travelling to the right,

$$v_y(x,t) = Re\left(v_y(\omega,k)e^{i(kx-\omega t)}\right)$$

Plugging the complex exponential into the equation of motion, we can find the relation between ω and k ,

$$k = \sqrt{i\,\omega\rho_0/\eta} = \left(\frac{1+i}{\sqrt{2}}\right)\sqrt{\omega\rho_0/\eta} = \frac{1}{\delta_V} + \frac{i}{\delta_V} \qquad \delta_V = \sqrt{\frac{2\eta}{\omega\rho_0}}$$

Applying the boundary condition at x = 0 we have,

$$v_y(x,t) = Re\left(v_y(\omega,k)e^{i(kx-\omega t)}\right) = V_{plate}\cos\left(\frac{x}{\delta_V} - \omega t\right)e^{-\frac{x}{\delta_V}}$$

The transverse wave dies off with a characteristic length δ_V , otherwise known as the *viscous penetration depth*,

$$\delta_V = \sqrt{\frac{2\eta}{\omega\rho_0}}$$

This demonstrates that we can't propagate transverse waves in a fluid. That requires a solid.

Sound attenuation in water

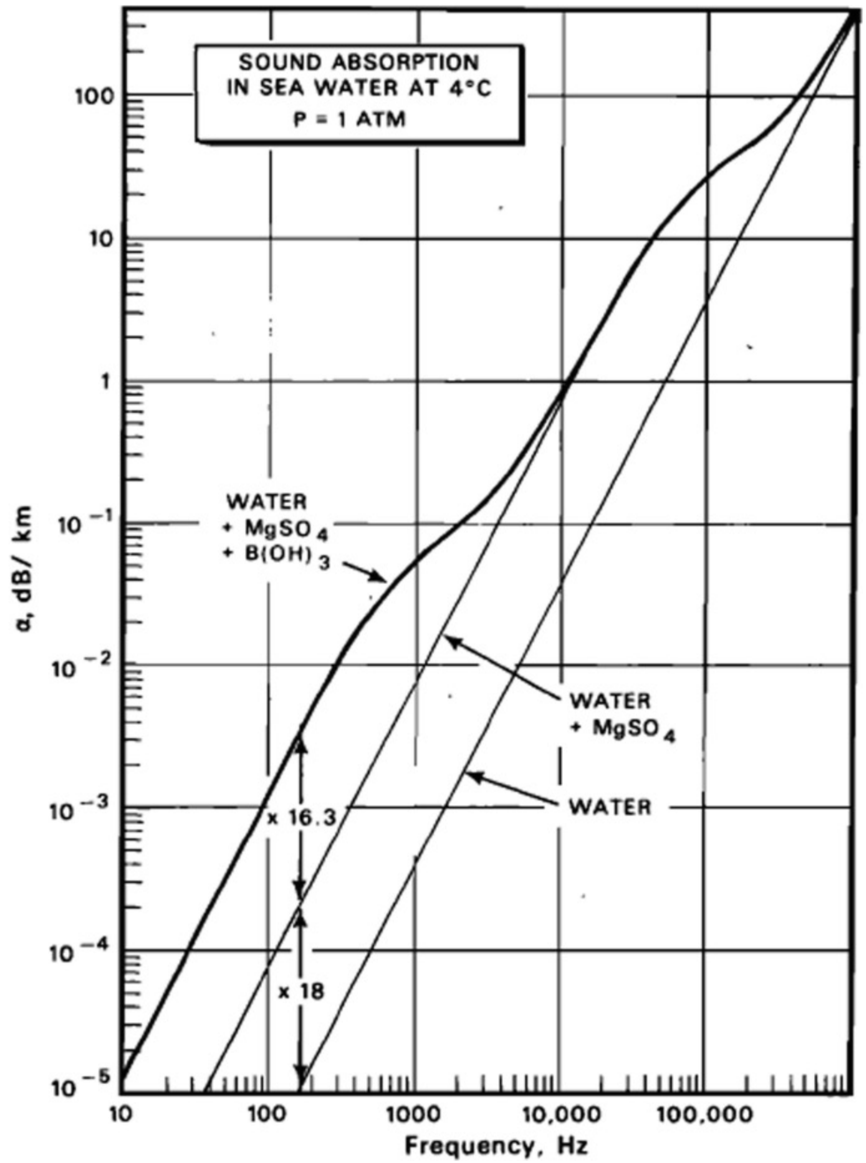
Okay, so transverse vibrations won't propagate in fluids but longitudinal sound waves do and they are attenuated for two major reasons (1) viscosity (2) internal degrees of freedom that are disturbed by the sound wave. The figure on the right illustrates the situation. Remember that the attenuation is defined by,

$$I(x) = I(0)\, e^{-\alpha x}$$

α is the length scale over which the sound wave loses energy and is damped out, somewhat like the viscous penetration depth. The plot shows α for pure water and sea water with two different compounds added. At low frequencies all three curves show a straight line. It's a log-log plot in which α increases by 100 for every factor of 10 in frequency f so the frequency dependence is quadratic,

$$\alpha = af^2$$

a is a constant depending on the material and the temperature. For pure water the f^2 dependence extends over many decades in frequency. That means low frequency sound waves travel *much* further in water than high frequency sound waves. That has implications for everything from submarines to whales. The bumps in the curves for water + $Mg(SO)_4$ and $B(OH)_3$ are another matter which we will take up shortly.



J. Lyman, R.H. Fleming, Composition of sea water. J. Mar. Res. **3**, 134–146 (1940), from S.L. , from Garrett, S.L. (2020). Attenuation of Sound. In: ***Understanding Acoustics***. Graduate Texts in Physics. Springer, Cham. https://doi.org/10.1007/978-3-030-44787-8_14

Sound attenuation by viscosity

As before, we'll assume the fluid velocity v_x varies only along the x-direction. The equation of motion *with viscosity* is now,

$$\frac{\partial v_x}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = \frac{\eta}{\rho_0}\frac{\partial^2 v_x}{\partial x^2}$$

The continuity equation stays the same,

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v_x}{\partial x} = 0$$

We can also use the relationship between changes in density and pressure,

$$p' = c_S^2 \rho'$$

These are linear equations so we can still assume the usual complex exponential solutions,

$$p', v_x, \rho' \sim e^{i(kx - \omega t)}$$

Plug these exponential solutions into the equation of motion and the continuity equation. We'll assume the part due to viscosity represents a small correction to the relationship between frequency and wavenumber. Doing that leads to,

$$k = \frac{\omega}{c_S} \frac{1}{\sqrt{1 - i\,\omega\eta / c_S^2 \rho_0}} \approx \frac{\omega}{c_S}\left(1 + i\,\omega\eta / 2c_S^2 \rho_0\right) = \frac{\omega}{c_S} + \mathrm{i}\frac{\eta\omega^2}{2c_S^3 \rho_0} = k_R + i\,k_I$$

This means that the pressure disturbance in the wave travels as a sound wave but with attenuation,

$$p' \sim cos(k_R x - \omega t) e^{-k_I x}$$

The attenuation constant α describes how the *intensity* changes with x. But the intensity is proportional to the square of the pressure *p*' so

$$I \propto (p')^2 \propto e^{-2k_I x} = e^{-\alpha x} \quad \rightarrow \quad \alpha_{Viscosity} = 2k_I = \frac{\eta f^2}{4\pi^2 c_S^3 \rho_0}$$

The sound attenuation constant due to viscosity does indeed vary at the square of the frequency, as measured.

In air, we must also include thermal relaxation! See S.L. Garrett, *Acoustics*.

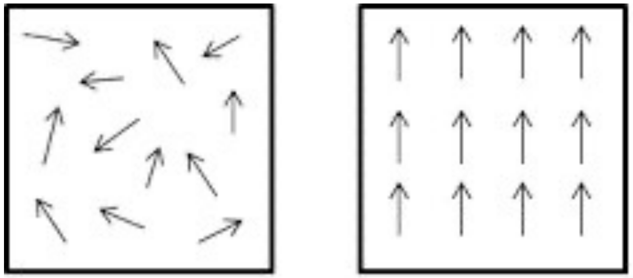
Relaxational sound attenuation

What about the bumps and the increased attenuation in the curves with $MgSO_4$ and $B(OH)_3$? In sea water there are two important chemical reactions taking place,

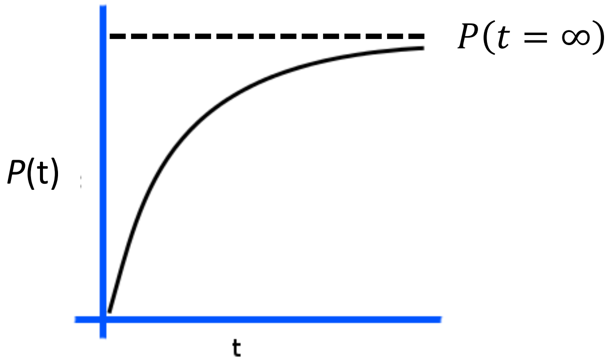
$$MgSO_4 + H_2O \leftrightarrow Mg^{3+} + SO_4^{2-} + H_2O$$

$$B(OH)_3 + (OH)^{-1} \leftrightarrow B(OH)_4^{-1}$$

Attenuation occurs because the sound wave comes along and temporarily disturbs the equilibrium of each chemical reaction. Once disturbed, the equilibrium takes some characteristic time τ to return to equilibrium. That leads to a widely observed type of frequency dependence. To illustrate I'll use an example familiar from E&M. Consider a collection of electric dipoles subjected to a sudden change in the electric field E at time t = 0.



$E = 0 \;\; P = 0 \;\; \Rightarrow \;\; E = E_0 \;\; P = \epsilon_0 \chi_0 E_0$



Before we apply the field the polarization *P* = 0. Now apply a step function electric field E_0. It takes some time for the dipoles to reach their new equilibrium. In the simplest case that's described by a single time constant τ :

$$\frac{dP}{dt} = \frac{(P(t = \infty) - P)}{\tau} \quad \rightarrow \quad P(t) = P(t = \infty)\left(1 - e^{-t/\tau}\right) \qquad P(t = \infty) = \epsilon_0 \chi_0 E_0$$

Here χ_0 is the permittivity in equilibrium after the electric field has been on for a long time. *P(t)* builds up just like the charge in an RC circuit subject to a step function in voltage. In that case the time constant $\tau = RC$.

Instead of a step function apply a *periodic* electric field at angular frequency ω and find the periodic polarization. As usual, represent periodic functions by,

$$P(t) = Re\big(P(\omega)e^{-i\omega t}\big) \qquad E(t) = Re\big(E_0\, e^{-i\omega t}\big)$$

Plugging the complex exponentials into the previous differential equation we get,

$$P(\omega) = \frac{\epsilon_0 \chi_0\, E(\omega)}{1 - i\omega\tau} = \epsilon_0\big(\chi_R(\omega) + i\chi_I(\omega)\big)E_0$$

The permittivity now has a real and an imaginary part, both of which depend on frequency,

$$\chi_R(\omega) = \frac{\chi_0}{1 + (\omega\tau)^2} \qquad \chi_I(\omega) = \frac{\chi_0 \omega\tau}{1 + (\omega\tau)^2}$$

Once we have the permittivity χ then we have the dielectric constant ϵ and the index of refraction *n* ,

$$\epsilon = \epsilon_0(1 + \chi) \qquad n = \sqrt{\epsilon/\epsilon_0} = \sqrt{1 + \chi_R + i\chi_I}$$

To find the attenuation assume we need the relationship between frequency and wavenumber *k*. Assume $\chi \ll 1$,

$$\frac{\omega}{k} = \frac{c}{n} \quad \rightarrow \quad k = \frac{\omega}{c}\sqrt{1 + \chi_R + i\chi_I} \approx \frac{\omega}{c}\left(1 + \frac{\chi_R}{2}\right) + i\frac{\omega}{2c}\chi_I = k_R + ik_I$$

From this you can see that a wave travelling through a dielectric like this would be attenuated:

$$E(x,t) = Re\big(E_0 e^{i((kx-\omega t))}\big) = E_0 \cos(k_R x - \omega t)\, e^{-k_I\, x}$$

The phase velocity and the attenuation constant of the wave now depend on the real and imaginary parts of the permittivity. Again, the intensity is proportional to the square of the field E so its attenuation constant is twice that of the field:

$$V_{Phase} = \frac{c}{k_R} = \frac{c}{1 + \chi_R/2}$$

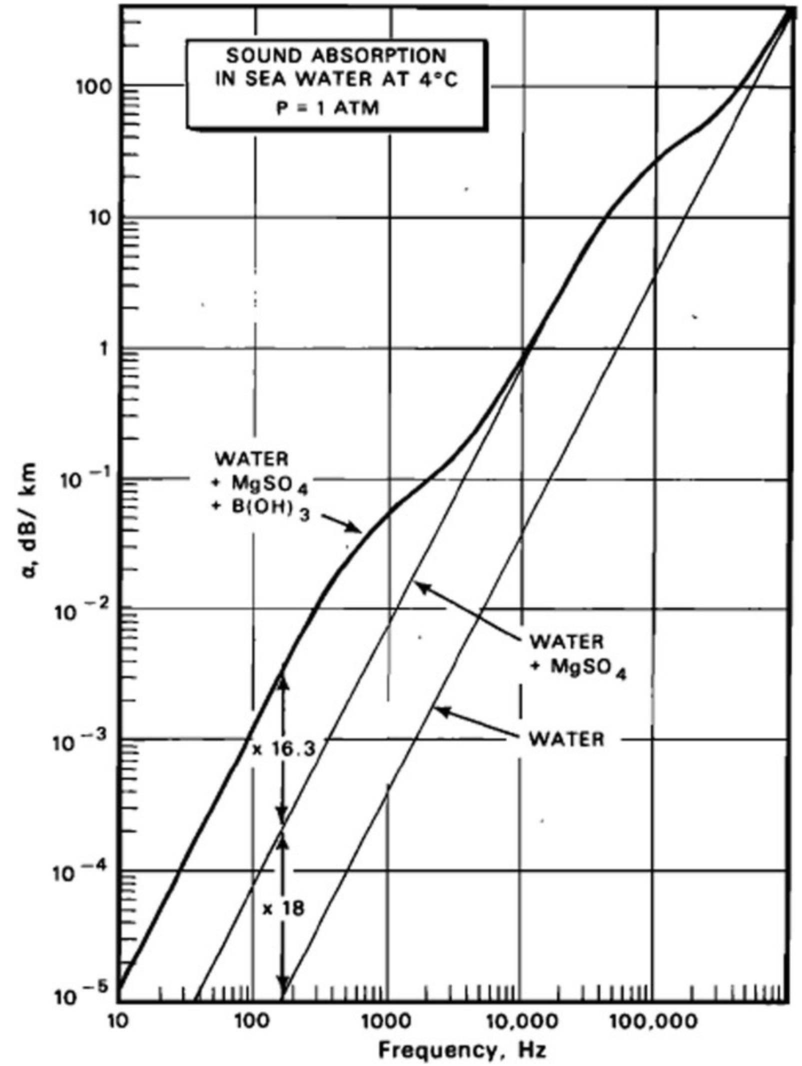
$$\alpha_{relax} = 2k_I = 2\frac{\omega}{c}\chi_I = \frac{2\omega^2\tau}{1 + (\omega\tau)^2}\frac{\chi_0}{c}$$

This kind of frequency dependence is characteristic of many relaxational processes. When $\omega\tau \ll 1$ the attenuation increases as ω^2. When $\omega\tau \approx 1$ it stops increasing and flattens off.

Returning to the case of sound waves, if there is some process like a chemical reaction going it, it takes some time τ to relax back to equilibrium. This leads to a time-dependent relationship between pressure and density, similar to the relationship between *P* and *E*. Relaxational attenuation will occur. In sea water, the $B(OH)_3$ reaction has a relaxation time of about $\tau \approx 10^{-4}$ sec . The sound attenuation stops increasing near $\omega\tau \approx 1$ which corresponds to a sound frequency of 1.6 kHz, which you can see on the log-log plot. For the $Mg(SO)_4$ reaction the time constant is about 10^{-6} sec. The $\omega\tau \approx 1$ condition occurs at a frequency of about 130 kHz, which you can also see on the plot.

There are similar attenuation features as sound passes through a diatomic gas like N_2 . There will be vibrational states of the molecule whose equilibrium occupations will be disturbed by a sound wave. Restoring the equilibrium occupation takes some time, again leading to attenuation with the form,

$$\alpha_{relax} = \frac{Af^2}{1 + (2\pi f\tau)^2}$$



J. Lyman, R.H. Fleming, Composition of sea water. J. Mar. Res. **3**, 134–146 (1940), from S.L. , from Garrett, S.L. (2020). Attenuation of Sound. In: ***Understanding Acoustics***. Graduate Texts in Physics. Springer, Cham. https://doi.org/10.1007/978-3-030-44787-8_14

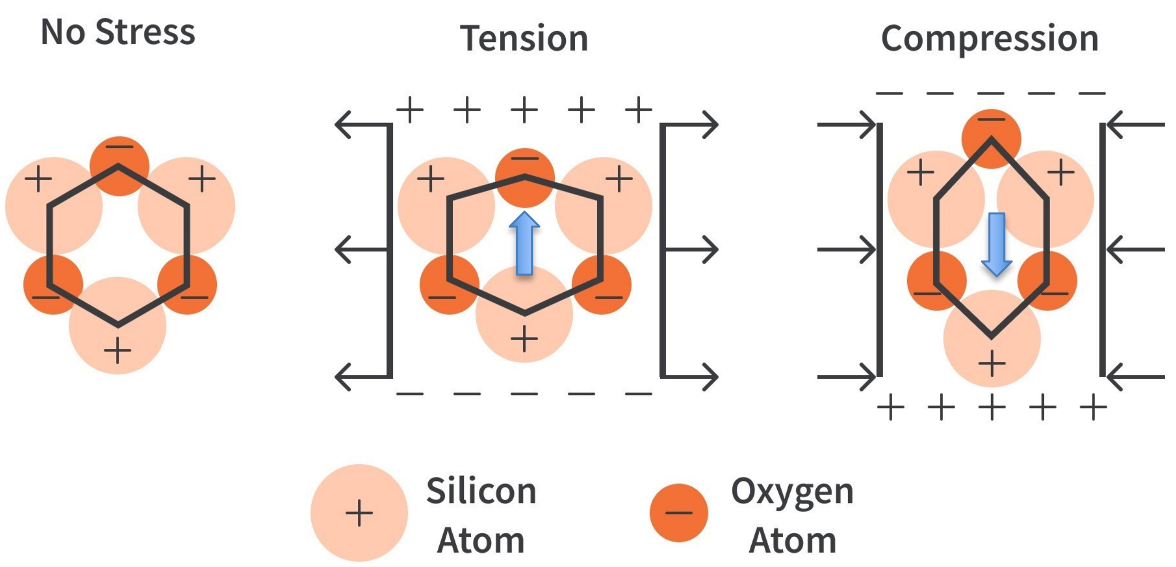
Piezoelectrics

The most widely used device for generating ultrasound is the piezoelectric solid - a material that changes size in response to an electric field. Piezoelectricity requires a special arrangement of atoms in the unit cell of the crystal. Quartz (SiO_2) is probably the best-known piezoelectric, though not the most efficient. The figure shows how tension and compression on a piece of quartz leads to an electric dipole moment (blue arrow) within the unit cell. In a macroscopic crystal of quartz, many unit cells are stacked on top of each other so the net result is a buildup of equal and opposite charge on the opposing crystal surfaces. The resulting electric field results in a detectable voltage between the crystal faces. The generation of ultrasound employs the *inverse* effect in which the application of a voltage between the two surfaces of the crystal causes it to shrink or expand. By applying a time-varying voltage the crystal can be made to vibrate at a particular frequency and generate sound waves.

No net dipole moment

Under tension, the centroid of (+) charge moves up while that of (-) charge moves down. A net dipole moment is created pointing up. This leaves a net (+) charge on the top surface and a net (-) charge on the bottom surface.

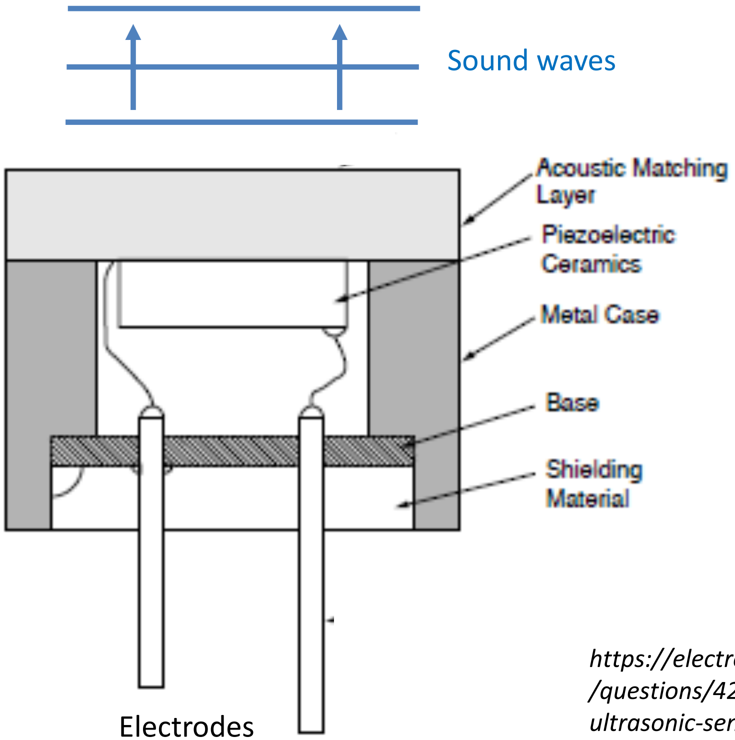
Under compression, the centroid of (+) charge moves down while the centroid of (-) charge moves up. A net dipole moment is created pointing down. This leaves a net (-) charge on top surface and net (+) charge on the bottom surface.



https://www.circuitbread.com/ee-faq/what-is-piezoelectric-effect

Transducers

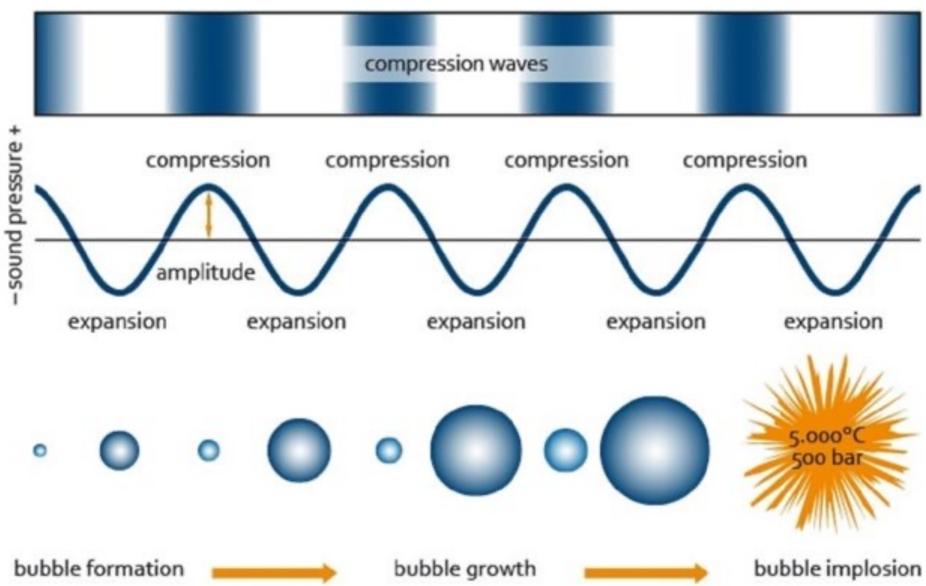
While quartz crystals are widely used in electronic circuits, lead zirconate, commonly referred to as PZT (chemical formula $Pb[Zr_xTi_{1-x}]O_3$ ($0 \le x \le 1$)) is the material of choice for ultrasonic transducers. That's because PZT expands and contracts much more than quartz for the same applied electric field. The figure below shows a very simple ultrasonic transducer. The two electrodes are electrically connected to the opposite faces of the piezoelectric material. Applying a voltage between them generates vibrations in the piezo which in turn generate ultrasonic waves in the medium of interest, similar to a loudspeaker. There is usually some kind of acoustic impedance matching layer on top of the piezo to maximize the amount of sound that is transmitted into the medium. Wavefronts for an outgoing sound wave are shown in blue. These waves go out, strike some surface and are reflected back. The resulting echoes return to the transducer, vibrate the piezo material and generate a voltage between the electrodes that is then sent to the electronics. You can buy one for a few dollars for simple tasks like range finding. The transducers used for medical imaging are much more sophisticated but it's the same basic idea.



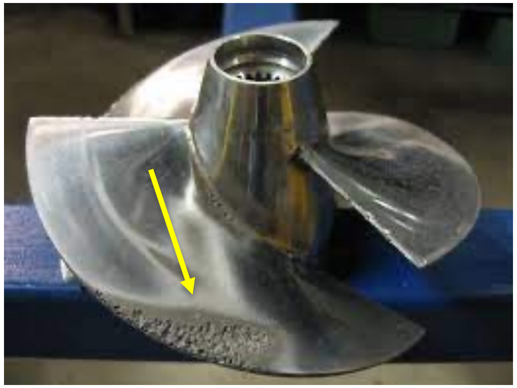
https://electronics.stackexchange.com/questions/426195/polarity-of-ultrasonic-sensor

Ultrasonic cleaning and cavitation

Ultrasound is used to clean everything from jewelry to silicon wafers. The actual cleaning is not done by the sound wave itself but through a process called *cavitation.* The object to be cleaned is placed in a bucket of liquid. PZT transducers generate large amplitude ultrasonic waves in the liquid at a frequency of about 40 kHz. A sound wave is a periodic string of high pressure and low pressure regions. In the low pressure regions it's thermodynamically favorable for the liquid to transform into a gas bubble or cavity, thus the name cavitation. This newly formed bubble then periodically expands and contracts and grows as shown. Eventually it reaches a critical size and implodes, generating very high temperatures and sending out shock waves. These shock waves dislodge the dirt from the piece to be cleaned. Unwanted cavitation can damage surfaces and ultimately destroy big things like ship propellers.

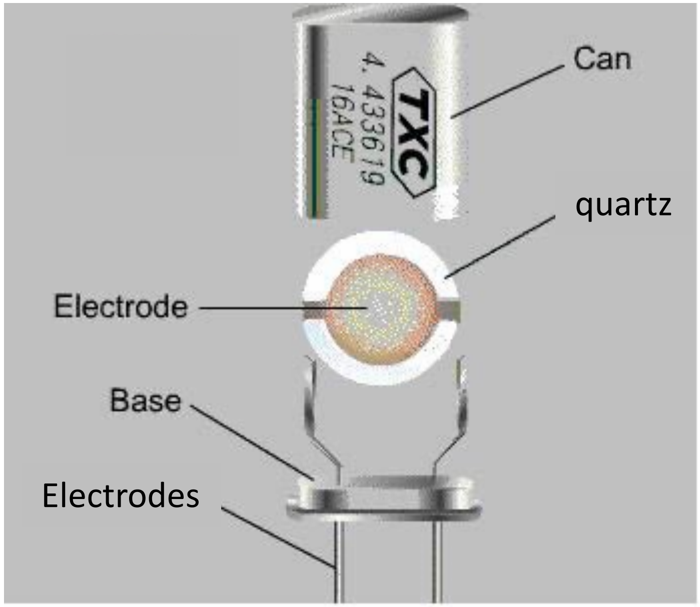


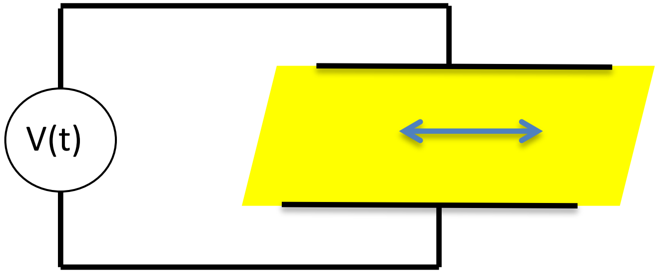
https://www.researchgate.net/publication/322552455_Sustainable_and_energy_efficient_leaching_of_tungsten_W_by_ultrasound_controlled_cavitation/figures?lo=1



Quartz crystal resonators

Electronic instruments that require a "clock" signal whose frequency is extremely stable use piezoelectric quartz crystals like the one shown below. There is no significant sound wave radiated away. Instead, the sound forms a standing wave at a sharply defined frequency that depends on the crystal dimensions. The most useful sound waves for this application are transverse waves that propagate between the top and bottom faces. The crystals are cut and polished so a sinusoidal voltage applied between the electrodes generates, via the piezoelectric effect, a back and forth *shear* displacement of the quartz. Quartz crystals for electronic instrumentation typically resonate in the 5 - 100 MHz frequency range.

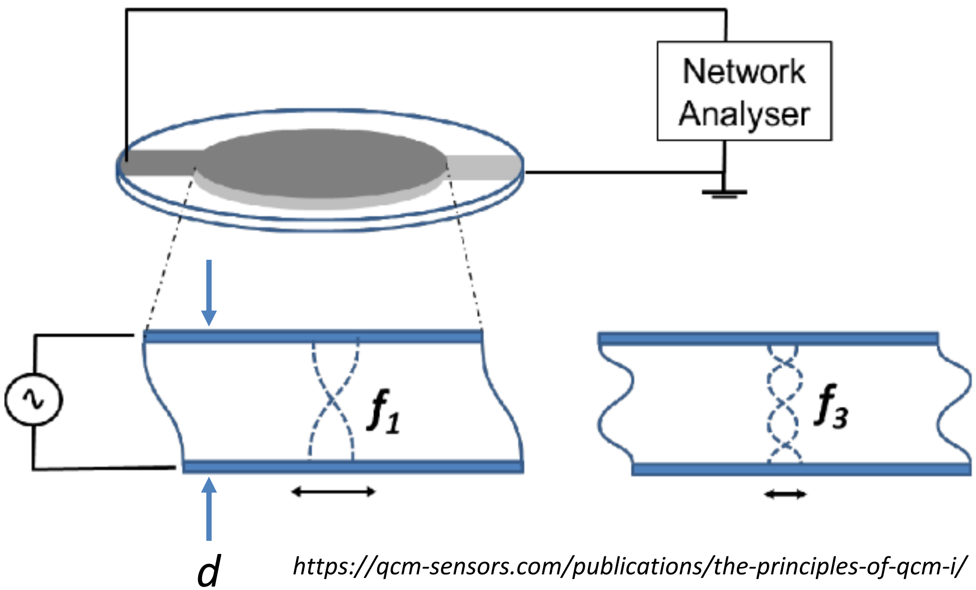




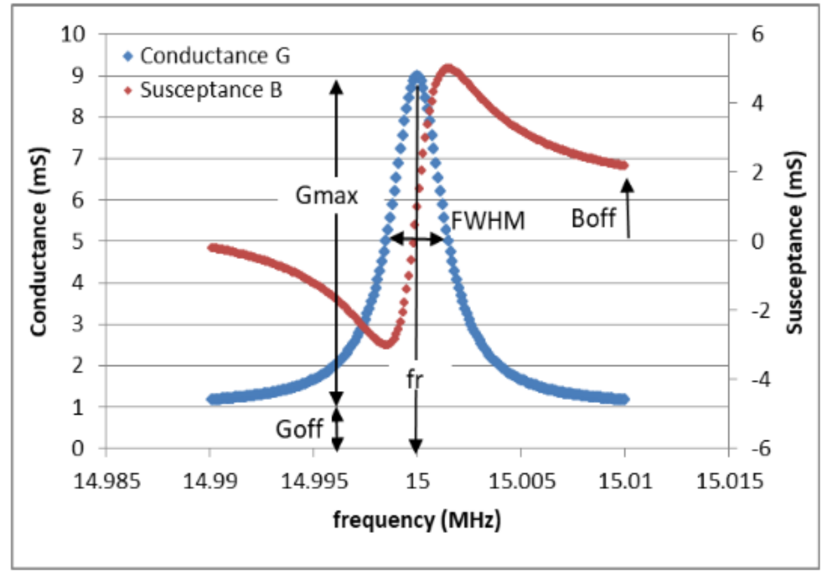
Shear-displacement of quartz crystal (yellow) in response to oscillating applied voltage. The material displacement is horizontal, in the direction of the blue arrow. The direction of sound wave propagation is vertical, between the top and bottom plate.

The next figure shows two different standing wave patterns within a quartz crystal. The wave labelled f_1 is the fundamental and corresponds to a sound wave in which the thickness *d* of the crystal corresponds to ½ wavelength. The pattern labelled f_3 corresponds to a harmonic in which *d* = 3/2 wavelengths. The frequencies are given by,

$$f_1 = \frac{c_s}{\lambda} = \frac{c_s}{2d} \qquad f_3 = \frac{c_s}{\lambda} = \frac{c_s}{(2d/3)} = 3f_1 \qquad c_s = sound\ velocity\ in\ quartz\ \approx 5000\ m/sec$$



https://qcm-sensors.com/publications/the-principles-of-qcm-i/



The electromechanical response of a quartz crystal resonator can be measured by a network analyzer. This instrument applies a sinusoidal voltage between the top and bottom faces of the crystal, generating standing sound waves via the piezoelectric effect. The network analyzer then measures the sinusoidal current that flows in response. For most frequencies the crystal just acts like an ordinary capacitor. But when the frequency is *very* close to f_1 (or f_3) the crystal responds like any resonant device. Standing sound waves build up and lead to a very sharp change in the current, as measured by the quantities *G* and *B* which depend strongly on frequency. The relation between applied voltage and current is given by,

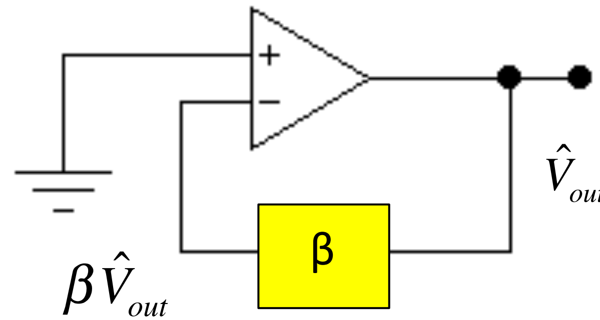
$$V(t) = \cos\omega t \qquad I(t) = G(\omega)\cos\omega t - B(\omega)\sin\omega t$$

The sharpness of any resonance is measured by the quality factor Q = f_r/FWHM (full width at half maximum). For the resonance shown in the figure Q is about 6000. (For comparison, a typical inductor-capacitor resonator might have a Q of 50.)

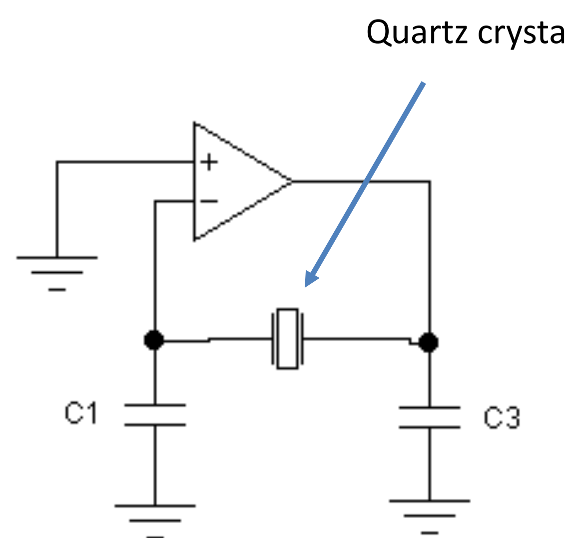
Quartz crystal oscillators

Recall our previous discussion of amplifiers and feedback. If the amplifier gain and feedback fraction satisfy the *Barkhausen criterion*,

$$A_V \beta = -1$$



then the circuit will oscillate. The circuit on the lower right uses a quartz crystal and two capacitors as its feedback network. The extremely sharp resonance of the quartz crystal determines the oscillation frequency. The quartz is cut in an orientation so that its dimensions don't change much with temperature, making the oscillation frequency very stable. Even a simple circuit like this will maintain a frequency of a few MHz to within ± 1 Hz or better despite typical room temperature variations. For higher stability the crystal is enclosed in a temperature-controlled box. This approach can lower the drift by several orders of magnitude. Virtually every piece of modern electronics contains quartz crystal oscillators like this.



Medical Ultrasound

The basic scheme of ultrasonic imaging involves the generation of ultrasonic waves from a transducer on the skin, letting these waves penetrate the body, bounce off parts of interest, capture the reflected waves and turn it all into an image. As we saw earlier, the reflection coefficient for a sound wave going from medium 1 into medium 2 is proportional to the mismatch of acoustic impedance $Z_2 - Z_1$. The more acoustic mismatch the better the image contrast between, for example, muscle and bone. The table shows the acoustic impedance for various body materials.

https://www.livescience.com/32071-history-of-fetal-ultrasound.html

Material	Velocity ($m \cdot s^{-1}$)	Acoustic impedance (Rayls)
Air	333	400
Fat	1446	1400000
Water	1480	1500000
Blood	1566	1650000
Muscle	1600	1700000
Bone	5000	7500000

Acoustic impedance =
sound wave pressure ($N \cdot m^{-2}$) / particle velocity ($m \cdot s^{-1}$)
Acoustic impedance is measure in Rayls
1 Rayl = 1 ($N \cdot m^{-2}$) / ($m \cdot s^{-1}$) = 1 $kg \cdot m^{-2} \cdot s$

https://www.google.com/search?q=acoustic+impedance&tbm=isch&sa=X&ved=2ahUKEwjhxbLp04v_AhXcm4kEHTcsDpgQ0pQJegQIDRAB&biw=1503&bih=961&dpr=2#imgrc=wuR672JYwoMZ0M

Why use ultrasound as opposed to lower frequency sound waves? If you recall from E&M, image resolution is ultimately limited by diffraction so the minimum discernable spot size is close to the wavelength so the higher the frequency, the better the image resolution. For example, a 3 MHz sound wave passing through water, which constitutes much of the body, has a wavelength of about 0.5 mm so that sets the approximate resolution. However, there is a tradeoff. As the frequency rises, sound waves are more rapidly attenuated as they pass through the body and attenuation rapidly degrades the signal. As a result, medical ultrasound is performed in the 1-20 MHz frequency range. Ultrasound is now used not only for imaging but also therapeutically. Since sound waves can be focused with millimeter resolution they can be used, somewhat like an ultrasonic cleaner, to physically change tumors and other biological features.

The requirements on the transducer for medical ultrasound are much more exacting than for ultrasonic cleaning. First, it's important to maximize the transmission of the sound as it moves from the solid transducer into the body and then back out again. Going back to acoustic impedances, recall that the energy transmission coefficient is given by,

$$\frac{I_{transmitted}}{I_{incident}} = \frac{4\, Z_1 Z_2}{(Z_1 + Z_2)^2}$$

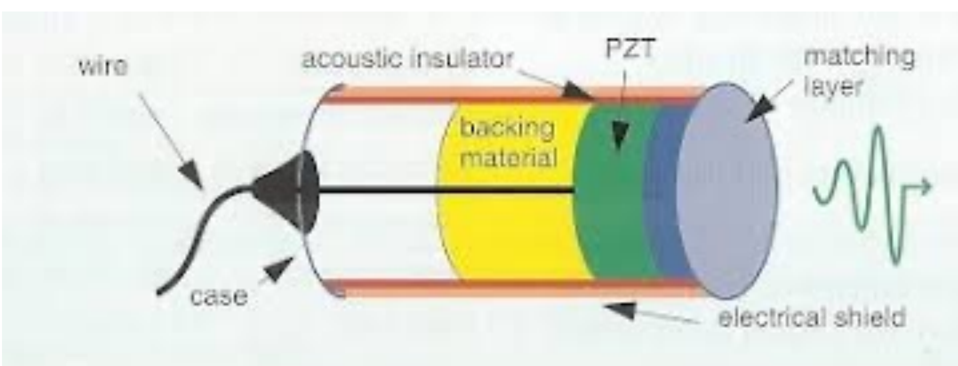
If, say $Z_1 = 5\, Z_2$ (e.g., sound going from the PZT transducer into water) then this ratio is 0.44 so more than half the sound energy available from the transducer never makes it into the body. Using a result from transmission line theory (relevant to both electromagnetic and sound waves) we can make the energy transfer 100% efficient by inserting an *impedance matching* layer. This layer must be made of a material whose acoustic impedance is the geometric mean of Z_1 and Z_2,

$$Z_{match} = \sqrt{Z_1 Z_2}$$

In addition, the thickness of the matching layer must be one quarter of the sound wavelength corresponding to the matching material,

$$L_{match} = \frac{\lambda_{match}}{4} = \frac{c_{s\,match}}{4\,f}$$

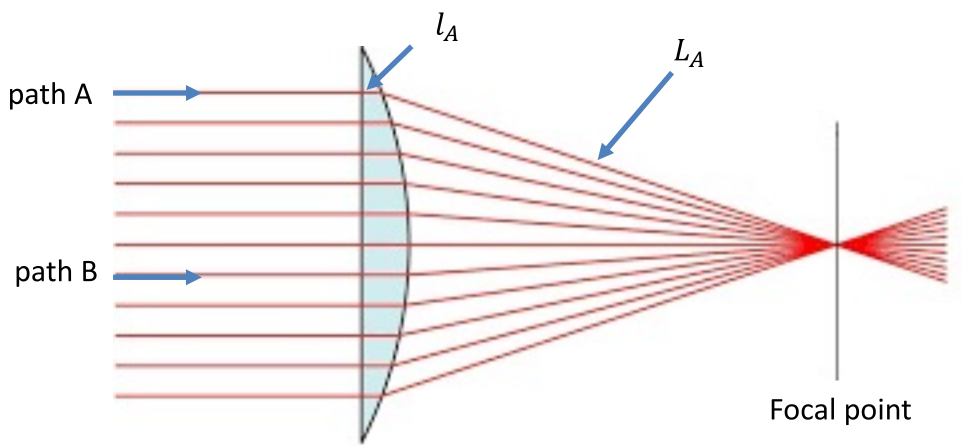
The overall transducer is shown below with the PZT, the matching layer and some *backing* material. That's generally some kind of composite whose acoustic impedance is close to the PZT but which rapidly attenuates sound waves. The aim is to keep the PZT transducer from "ringing" too long after it's excited. Unlike the situation with quartz crystals discussed previously, we don't want extremely high Q transducers for transducers. High Q implies excessive ringing after the electrical excitation and that makes it difficult to both shape the outgoing bursts of sound and to process the incoming echoes that are used to form the image.



https://quizlet.com/599300794/chapter-8-transducers-flash-cards/

Medical transducers are now made from a large array of PZT transducers, each of which can generate outgoing sound waves and then detect the echoes that come back. It's worth recalling how an ordinary lens focuses light. Suppose the left side of the lens is illuminated by an incident beam. Before reaching the lens the incoming light wave has a constant phase along a wavefront perpendicular to the beam. Now look at light moving along path A. It passes through glass whose thickness is l_A and then travels an additional distance L_A to reach the focal point. The total phase change is given by,

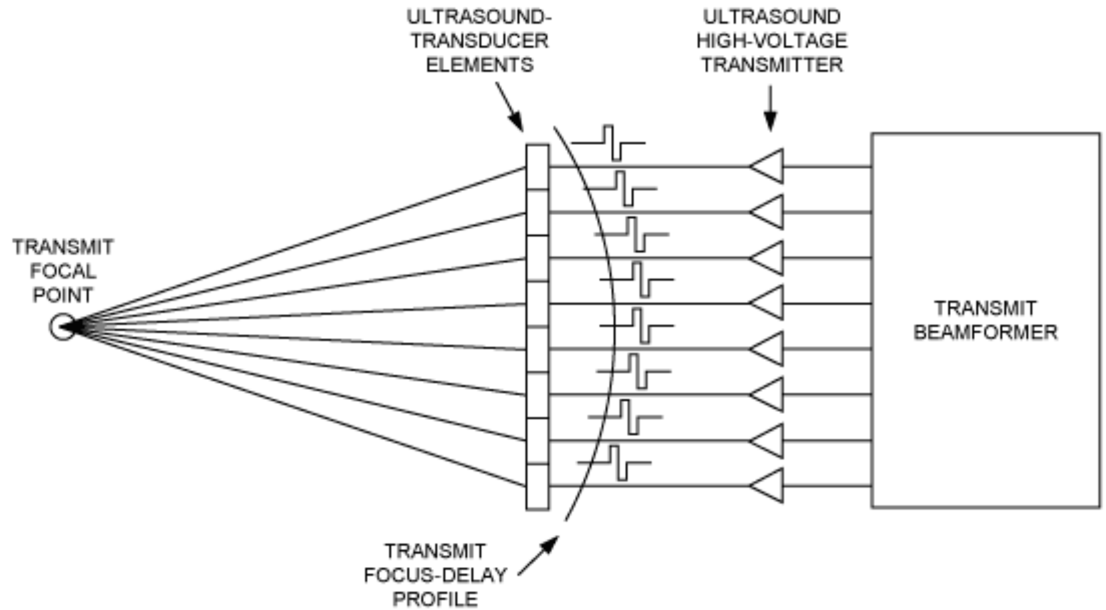
$$\phi_A = \phi_{glass}^{A} + \phi_{air}^{A} = k_{glass} l_A + k_{air} L_A = \frac{\omega}{c_{glass}} l_A + \frac{\omega}{c_{air}} L_A$$



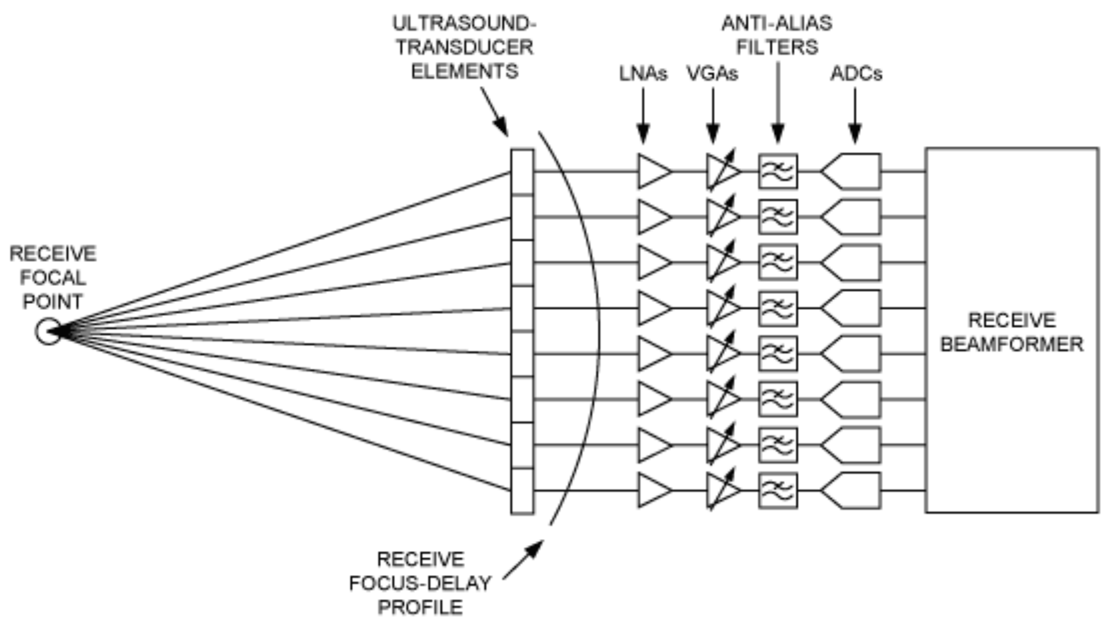
A wave moving along path B will have its phase changed but now l_B is larger and L_B is smaller than for path A. So along path B the wave picks up a phase,

$$\phi_B = \frac{\omega}{c_{glass}} l_B + \frac{\omega}{c_{air}} L_B$$

For the lens to focus, the phase must be the same for *all* paths to the focal point. In a conventional lens this problem is solved by grinding the lens to the correct curvature. That sort of thing can be done with sound waves but a much more versatile way to focus sound waves is to construct the lens as an *array* of many small PZT transducers. Then each transducer can be separately excited electronically. To ensure that all the sound waves have the same phase, the transducers are excited at different *times*. In effect, the phase shift produced by a conventional lens is replaced by a phase shift in the radio frequency signal sent to excite the piezoelectric element. Since the frequencies are very low (a few MHz) compared to optical frequencies (10^{14} Hz) , this kind of precise electronic control is possible. This approach is generally termed *phased array* beam manipulation. It's also widely used for electromagnetic waves in radar. The overall scheme is shown on the next slide.



Tutorial 4038, Maxim Integrated Circuits



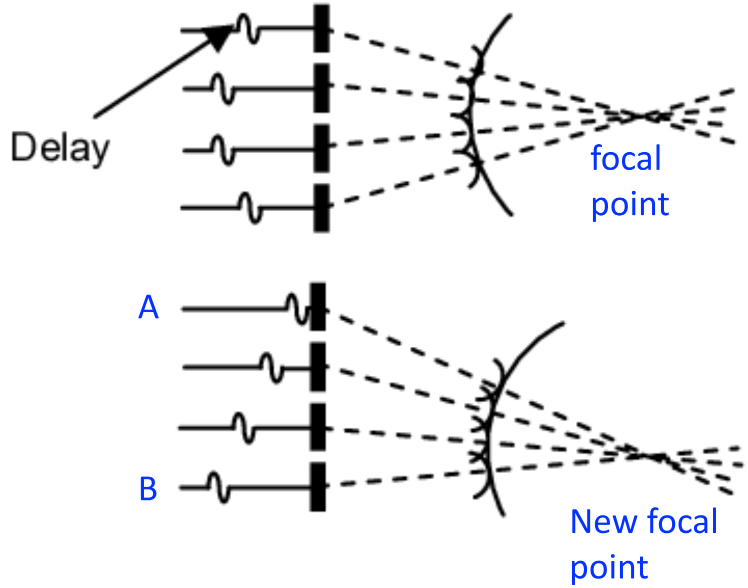
Transmitting the focused beam

The top figure shows how voltage pulses with differing delay times will excite the array of transducer elements to generate a focused beam of ultrasound. And by changing the pattern of delay times the focal point can be moved to different points in the body.

Receiving the echo

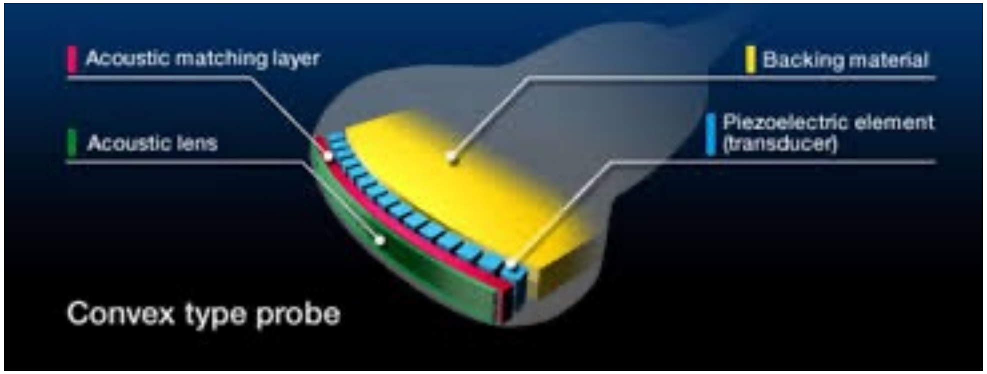
Sound bouncing off an object located at the focal point now returns to each transducer. The received sound disturbance from each transducer generates an electrical impulse that is (1) amplified (LNA – low noise amplifier) (2) has its amplitude adjusted for loss of energy as it passes through the body (VGS) (3) filtered to keep out unwanted signals and (4) digitized in an analog to digital converter (ADC). Since the reflected wave from the focal point arrives at each transducer at a different time, the digitized signals can be suitably corrected to all arrive at the same time, just as a lens would do. Next, this focal point must be scanned over the region of interest.

Phased array of ultrasonic transducers



In order to form a complete image, the reflected sound intensity must be recorded at each point in the body region of interest. That's done by varying the time delays to move the focus around electronically. For example, to focus the beam in the bottom picture, the transducer at A would need to be excited earlier than the transducer at B. These same delays would then be used to correct the sound wave reflected from the new focal point.

https://www.semanticscholar.org/paper/High-intensity-ultrasound-phased-array-for-surgical-Tan-Chu/2d2f8e3140f7189774f4f2d8263da223943aa2c4



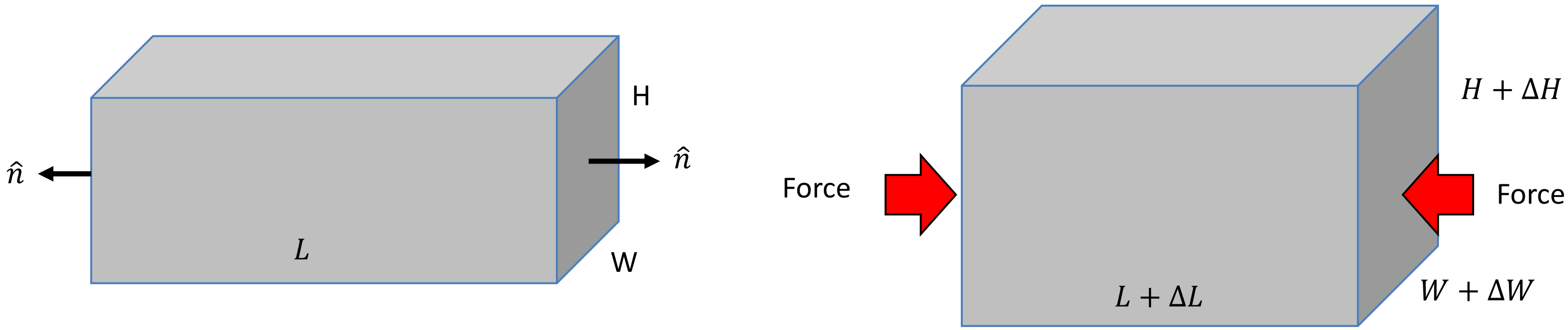
https://europepmc.org/backend/ptpmcrender.fcgi?accid=PMC5022373&blobtype=pdf

The figure shows one type of modern ultrasound transducer array. The individual elements are often arrayed on a more convex surface with a layer of varying thickness that acts like an acoustic "lens". These features may optimize the ability of the device to focus signals from the transducer array.

The ability to digitize the returning echo signals opens up other imaging possibilities. For example, the returning sound wave may be *Doppler-shifted* if it reflects off a region of flowing blood. That Doppler shift will show up as a shift in frequency of the reflected wave relative to the transmitted wave.

Stress and Strain

To discuss sound waves in solids we need to talk about the relationship between force and displacement. Consider a rectangular block of material that is L long, W wide and H tall. Now apply a force uniform force *F* perpendicular to the end faces. The resulting deformation, greatly exaggerated, is shown on the right.



When we talk about materials in the *elastic* limit, we mean that the material obeys Hookes's law : the deformation is *proportional* to the force per unit area. Like a spring, when we squeeze an elastic body and then release the pressure the body returns to its original shape. For elastic bodies, Hooke's law takes the form,

$$Stress = \frac{Force}{Area} = \frac{F}{WH} = Y_m \frac{\Delta L}{L} = Y_m \cdot Strain$$

The quantity on the left is called the *stress* and the fractional deformation $\Delta L/L$ is called the *strain*. Why the *fractional* deformation $\Delta L/L$? Think about applying the same forces to a block whose initial length is 2*L*. Then ΔL will be be twice as large but $\Delta L/L$ remains the same. The quantity Y_m is called the Young's modulus. It's like a spring constant and is different for every material. To get the signs correct, note that the force on each end acts *inward* so $\vec{F} \cdot \hat{n} < 0$ on each end where $\hat{n}$ is the unit normal vector to the surface. The negative sign for the stress implies a negative sign for the strain so $\Delta L/L$ is negative.

Squeezing along the *L* direction causes the material to expand along the *W* and *H* directions. That's described by a second material-specific constant known as the Poisson ratio σ, defined by,

$$\frac{\Delta W}{W} = \frac{\Delta H}{H} = -\sigma \frac{\Delta L}{L}$$

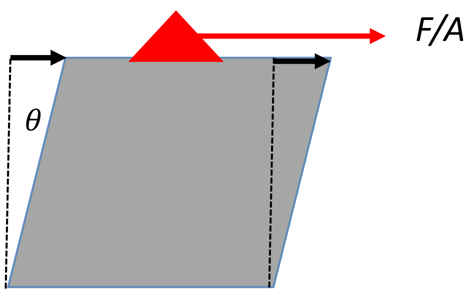
For isotropic materials like steel whose properties don't depend on direction Y_m and σ are all that's needed to characterize the elastic properties. For a single crystal we would need more parameters.

In an elastic material the response (the strain) is linearly proportional to the force per unit area (the stress). Linearity implies that superposition can be used to solve problems in elasticity, as in electricity and magnetism. For example, imagine that we apply a uniform pressure *P* on all 6 sides of the block by letting it fall to the bottom of a lake. We'll leave it as a homework problem to show, using superposition, that the fractional change in volume is given by,

$$\frac{\Delta V}{V} \approx \frac{\Delta W}{W} + \frac{\Delta H}{H} + \frac{\Delta L}{L} = -\frac{3(1-2\sigma)}{Y_m} P = -\frac{P}{K}$$

where *K* is called the *bulk modulus*. This equation also shows why the Poisson ratio $\sigma <$ ½. If the material had $\sigma >$ ½ then the block would expand under pressure and it would be possible to extract work from it. There would be no stable equilibrium situation.

In addition to compressing and expanding, solids may undergo *shearing* motion. Take the same block and glue it to the floor. Assume the top and bottom faces have area *A*. Now apply a sideways force per unit area = *F/A*. The block will distort in what is termed a pure *shearing* motion. Superposition can be used to show that the angle of distortion θ is given by,



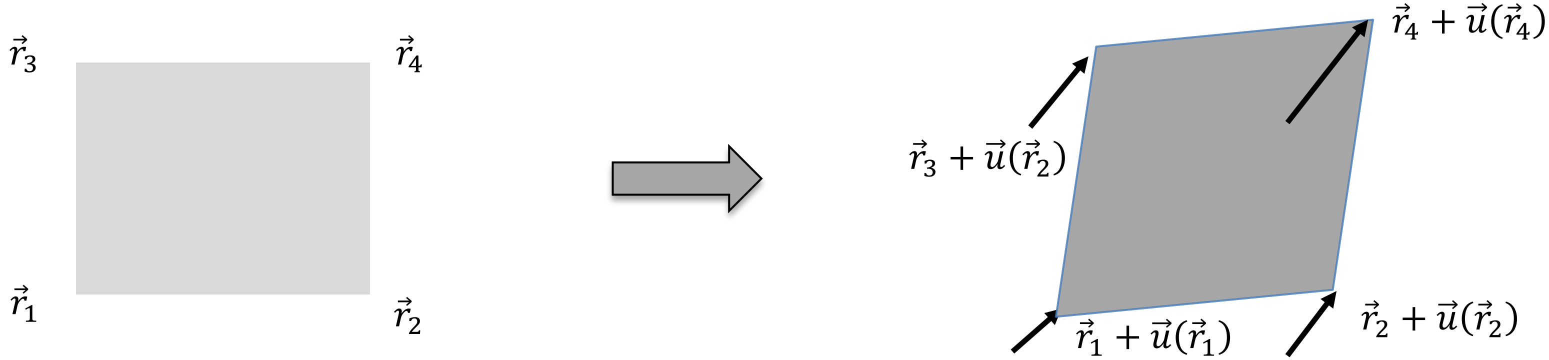
$$\theta = 2\frac{1+\sigma}{Y_m}\frac{F}{A} = \frac{1}{\mu}\frac{F}{A}$$

The quantity μ is called the *shear modulus*. Sound waves in solids may involve both shearing and compressional motions. To describe both it is useful to introduce another elastic constant λ (not to be confused with the wavelength!). These two new quantities are called the Lame constants and are related to the Young's modulus and Poisson ratio,

$$\mu = \frac{Y_m}{2(1+\sigma)} \qquad \lambda = \frac{\sigma Y_m}{(1+\sigma)(1-2\sigma)}$$

Local displacement field

For wave propagation through solids we'll focus on the local displacement $\vec{u}(\vec{r},t)$ of a tiny piece of solid from its equilibrium position $\vec{r}$. Consider a piece of the solid whose corners are located at equilibrium positions $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_4$. Now apply some stress to the solid so that it distorts. The corners now move to new positions labelled $\vec{r}_1 + \vec{u}(\vec{r}_1), \vec{r}_2 + \vec{u}(\vec{r}_2)$, etc. $\vec{u}(\vec{r},t)$ is the local displacement and it depends on space and time. However $\vec{r}$ does *not* depend on time. It's just a label for different equilibrium locations in the solid.



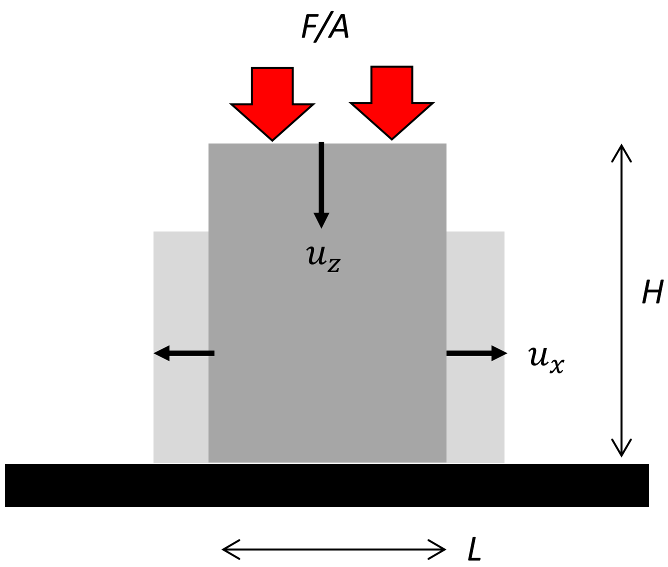
To connect with what we previously said about Hooke's law, again consider the block of height *H*, length *L* and width *W*. Imagine it rests on an immovable table. Now exert a force per unit area F_z/A where $F_z = -F$ and the area $A = LW$. The z-displacement of the top of the block is $u_z(z = H)$ is given by,

$$u_z(z = H) = Y_m H(F_z/A)$$

$$\frac{u_z(z = H)}{H} = Y_m(-F/A)$$

This is the just the fractional change in dimension introduced earlier, but in the language of local displacement u_z . u_z = 0 at z = 0 (the table top) and it varies linearly with z. This is called a uniform strain. Therefore the left side can be written as,

$$\frac{\partial u_z}{\partial z} = Y_m(F_z/A)$$



Sound waves in solids

Assuming the solid is elastic and isotropic, focus on the local displacement from equilibrium $\vec{u}(\vec{r},t)$. It requires some mathematics to prove, but the displacement field obeys the equation of motion,

$$\varrho \frac{\partial^2 \vec{u}}{\partial t^2} = (\lambda + \mu)\nabla\,(\nabla \cdot \vec{u}) + \mu\nabla^2\,\vec{u}$$

ρ is the density of the material (kg/m^3) and λ, μ are the two Lame constants. This looks somewhat like a wave equation in three dimensions but with a complicated middle term. Since it is linear the solutions can be written in the wavelike form,

$$\vec{u}(\vec{r},t) = Re\left(\vec{u}_k\, e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right)$$

All the space and time dependence is now contained in the exponential factor where $\vec{k}$ is the wavevector. The full solution can be written as the sum of a longitudinal displacement field $\vec{u}_P$ (parallel to $\vec{k}$) and and transverse displacement field $\vec{u}_S$ (perpendicular to $\vec{k}$.)

$$\vec{u} = \vec{u}_S + \vec{u}_P = Re\left(\vec{u}^P_{\vec{k}}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \vec{u}^S_{\vec{k}}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right) \qquad \vec{u}^P_{\vec{k}} \parallel \vec{k} \qquad \vec{u}^S_{\vec{k}} \perp \vec{k}$$

Plugging the expressions $\vec{u}^P_{\vec{k}}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ and $\vec{u}^S_{\vec{k}}\; e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ separately into the wave equation and using some vector calculus it is straightforward to show that the longitudinal waves have a phase velocity,

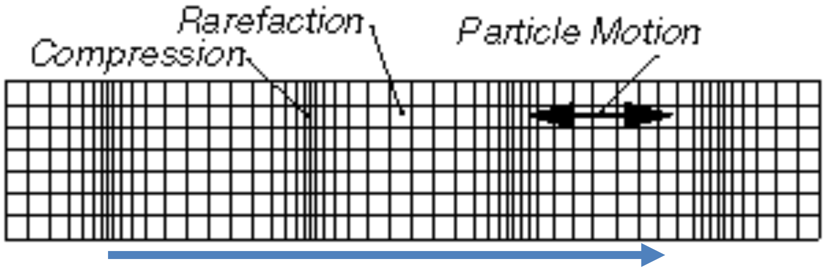
$$c_P = \sqrt{(\lambda + 2\mu)/\varrho} \qquad P - wave\;\;(longitudinal)$$

while the transverse waves have a smaller phase velocity, given by,

$$c_S = \sqrt{\mu/\varrho} \qquad S - wave\;\;(transverse)$$

Sound velocities in solids are typically 5-6 x 10^3 meters/sec, which is much higher than in liquids. Recall that in water, the sound velocity is about 1500 m/sec.

The figure below shows the compressions and expansions that occur in a longitudinal sound wave. In seismology these are called *P-waves* or principal waves.

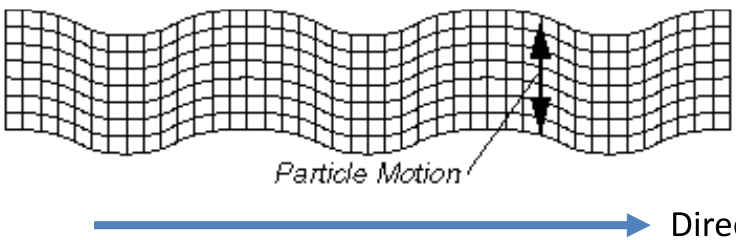


Longitudinal (P) wave

Direction of propagation

https://www.ux1.eiu.edu/~cfjps/1300/earthquake.html

The transverse waves have displacement perpendicular to the direction of propagation, as shown below. Seismologists call these S-waves because they involve a *shearing* motion. Our analysis is for an isotropic solid whose elastic properties are the same in all directions. In that case there is only one shear-wave velocity. However, crystals are not always isotropic, in which case the shear wave velocity depends on the direction of the shear displacement vector.

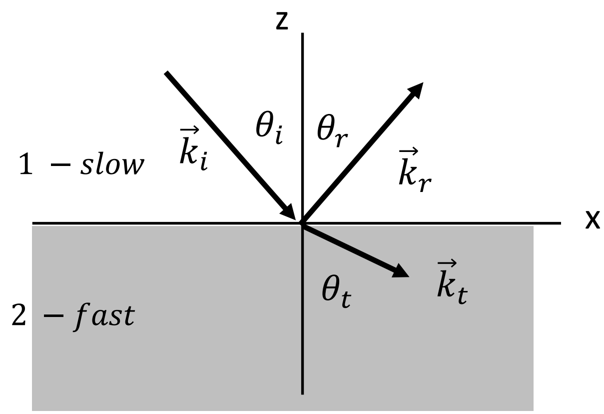


Shear (S) wave

Direction of propagation

https://www.ux1.eiu.edu/~cfjps/1300/earthquake.html

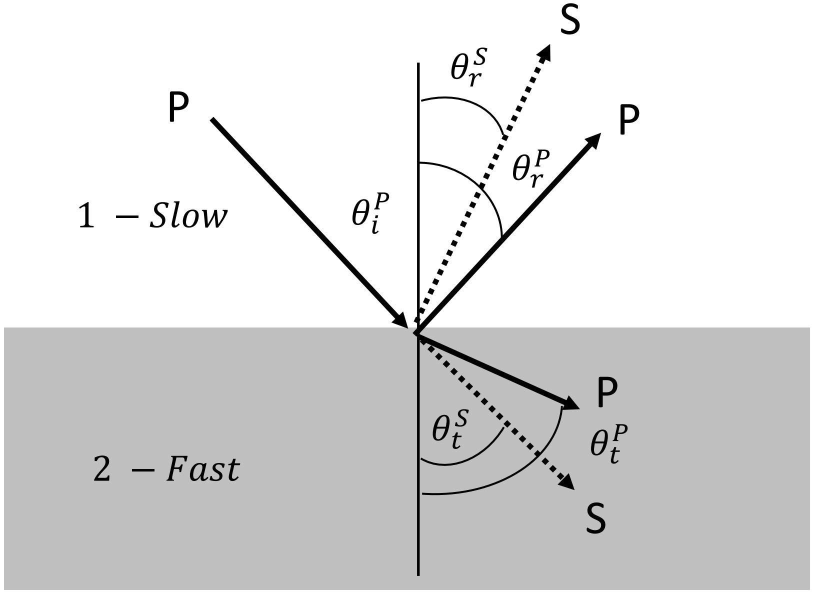
Snell's Law



Sound waves also obey Snell's law. For fluids the derivation goes exactly the same way as it does for electromagnetic waves. Let the incident, reflected and transmitted waves vary as $\cos(\omega t - k_i \cdot \vec{r})$, $\cos(\omega t - k_r \cdot \vec{r})$ and $\cos(\omega t - k_t \cdot \vec{r})$ respectively. The three wavevectors are shown. Since our equations are all linear, no new frequencies are generated so ω is the same for all three waves. The three waves must be in phase at all times along the boundary between the two media, defined as the plane z = 0. If $\vec{r}$ is any vector in the plane z = 0 this implies that $k_i \cdot \vec{r} = k_r \cdot \vec{r} = k_t \cdot \vec{r}$. Trigonometry then implies that $\theta_i = \theta_r$ and $k_i\, sin\theta_i = k_r \sin\theta_t$. The wavenumber is related to the speed of sound through $k = \omega/c$ so we arrive at Snell's law,

$$\frac{1}{c_1} sin\theta_i = \frac{1}{c_2}\sin\theta_t$$

As with light waves travelling from glass (slow) into air (fast), sound waves passing from the slower medium to a faster medium are refracted *away* from the normal.



In solids, this picture is complicated by the presence of both P and S waves. Consider the same picture where the solid arrows correspond to P waves and the dashed arrows to S waves. An incoming P wave strikes the boundary and generates reflected and transmitted P *and* S waves. Fortunately Snell's law for this more complicated situation is still easy to prove,

$$\frac{sin\theta_i^P}{c_{P1}} = \frac{\sin\theta_t^P}{c_{P2}} = \frac{\sin\theta_r^S}{c_{S2}} = \frac{\sin\theta_t^S}{c_{S2}} \qquad \theta_i^P = \theta_r^P$$

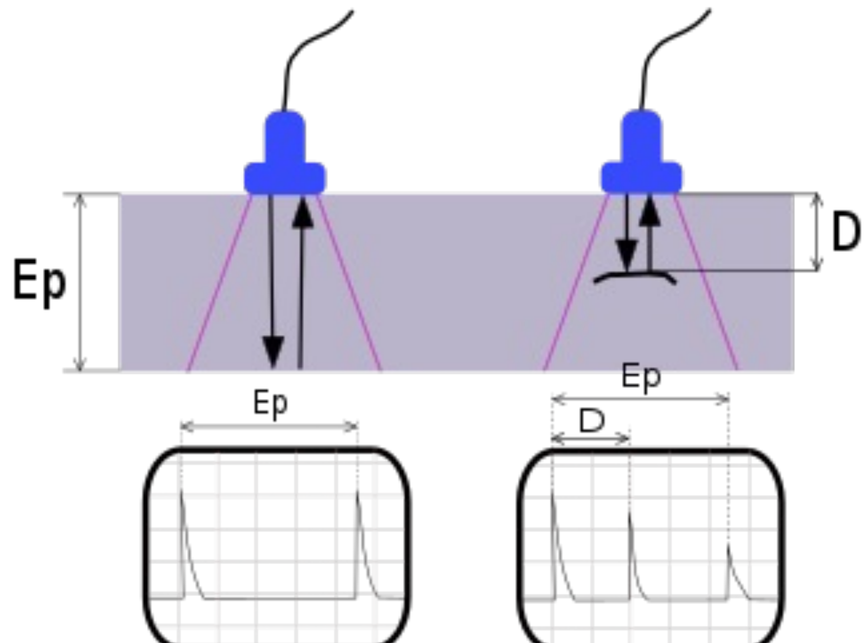
(See *Unearthing Fermi's Geophysics*, G. Segre and J. Stack, U. Chicago Press):

https://www.sealaviation.com/about-non-destructive-testing

Non destructive testing

Ultrasonic waves in solids are widely used to detect flaws, particularly in metal parts. This is known as non-destructive testing. Frequencies up to about 10 MHz are commonly used. Referring to the figure below, if there's a flaw some distance D beneath the surface then that produces an echo as shown on the screen. Welds are a particular concern and ultrasound is routinely used to test them in critical application like pipelines. Sometimes the flaws generate higher harmonics which can be useful for detecting corrosion in places where it's not visible on the surface.

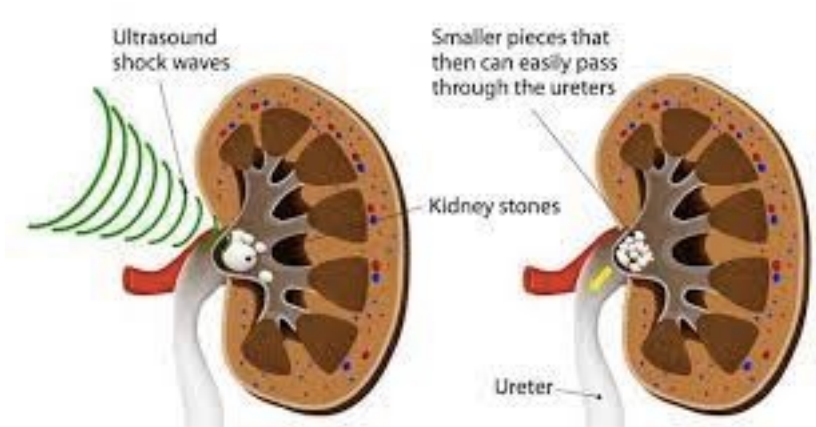
https://en.wikipedia.org/wiki/Ultrasonic_testing



https://en.wikipedia.org/wiki/Ultrasonic_testing

Destructive ultrasound

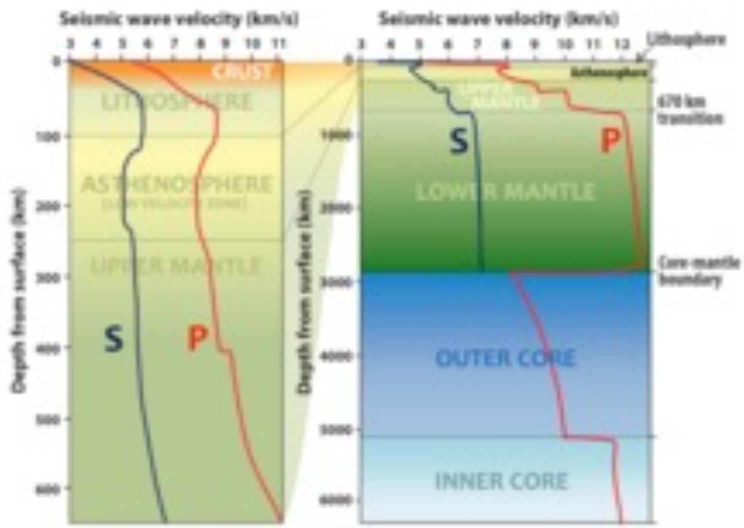
In some cases ultrasound can be focused to sufficient intensity to generate shock waves. These can be used to break up kidney stones.



https://www.urologysanantonio.com/kidney-stones/lithotripsy

Seismology

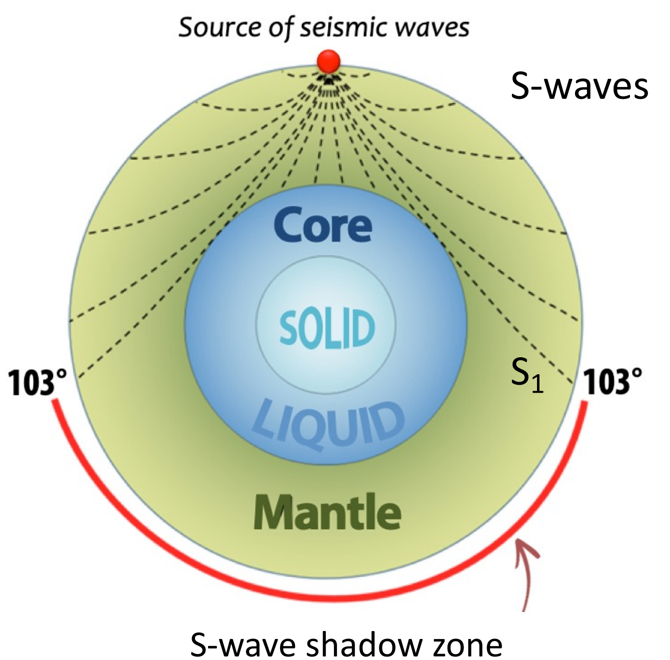
Sound waves in solids are central to seismology. What is known about the earth's structure has come largely from the study of seismic waves. Aside from some variations nearer the surface, The figure below shows that as we go down into the earth both S and P waves travel faster. Then, at a depth of 3000 km, S-waves cease to propagate and the P- wave sound velocity drops significantly. Since shear waves do not propagate through a liquid, the conclusion is that the earth's outer core is a liquid. The next figure shows how observations bear out this hypothesis. Sound waves are shown radiating out from the epicenter of an earthquake and passing through different regions of the earth.

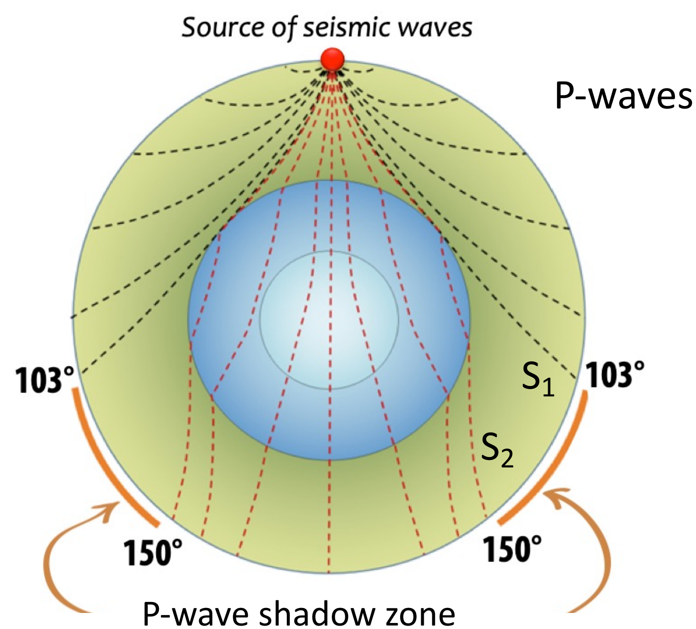


https://en.wikipedia.org/wiki/Seismic_wave

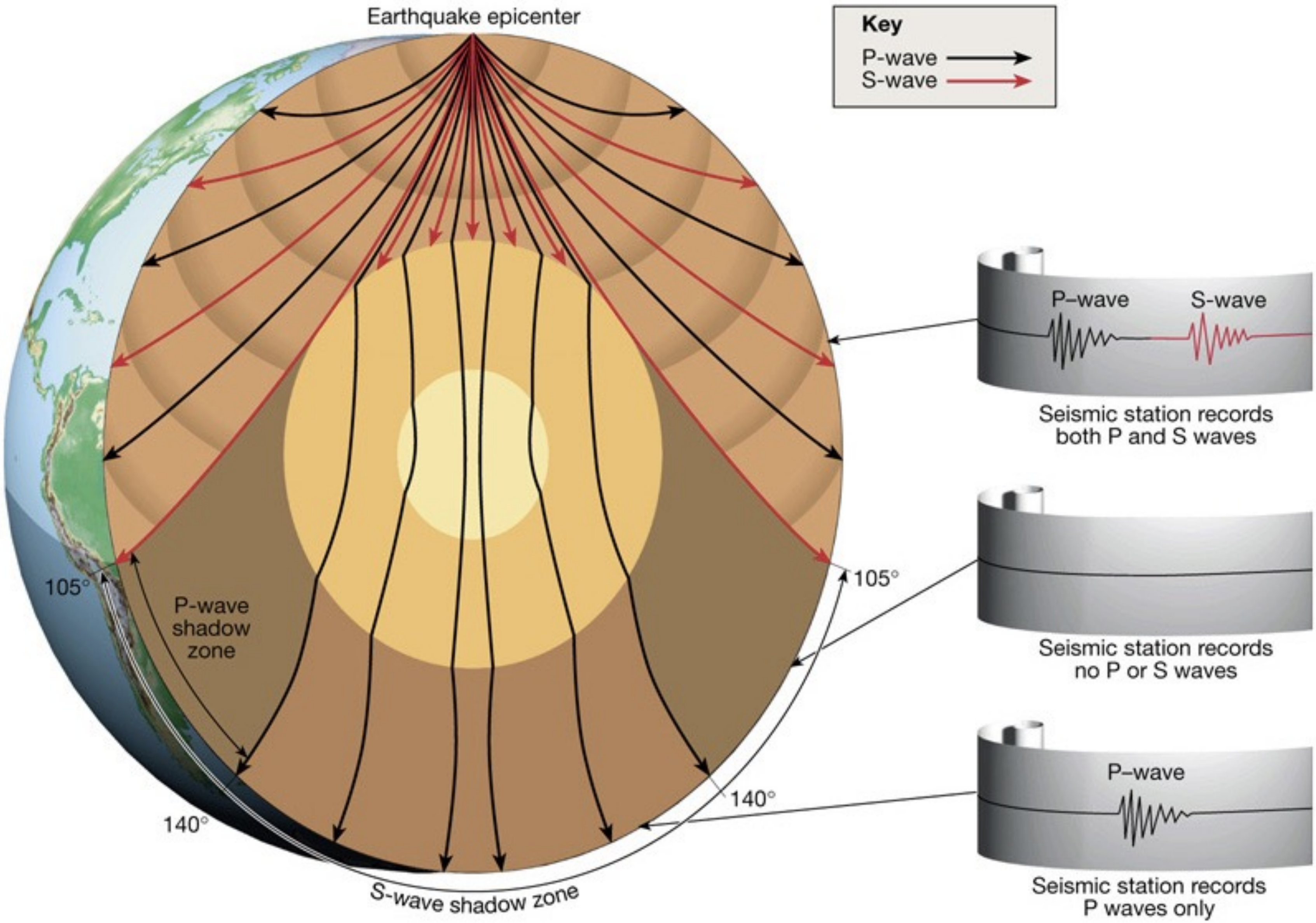
Focus first on the S-waves. If the sound velocity were the same everywhere the rays would simply point radially outward from the epicenter and never change direction. However, as the wave goes deeper into the earth it speeds up. The rays bend *away* from their initial direction, curving around and eventually returning to the surface. Notice the large S-wave *shadow zone* where no S-waves are detected. Ray S_1 reaches the Mantle-Core boundary with $\theta_i^S = \theta_r^S = 90°$. It just skims the core and continues on, reaching the earth's surface at $103°$ latitude. To observe S-waves at latitudes larger than $103°$ they would need to penetrate the core. Evidently they don't, which was key to realizing that the earth's outer core is a liquid.

P-waves, which are longitudinal, *can* travel through the liquid outer core and make it all the way to the opposite side of the earth. Take a look at the ray S_1 that just skims the outer core. Like the corresponding S-wave, its angle of incidence is $90°$. Part of the ray reflects and continues on to reach the surface at $103°$ latitude. But part of it, labelled S_2, refracts *into* the liquid core since the P-wave velocity in the liquid is lower than in the solid mantle. S_2 travels through the liquid core, then back into the mantle, eventually reaching the surface at $150°$. In between $103°$ and $150°$ there are no P-waves, just a P-wave shadow zone. The earth's *inner* core turns out to be solid, as hypothesized by Inge Lehmann in 1936. She noticed that seismic data showed evidence for very *faint* P-waves in the P-wave shadow zone. She interpreted these signals as P-waves from the liquid outer core reflecting off the inner core, which is consistent with a solid inner core.





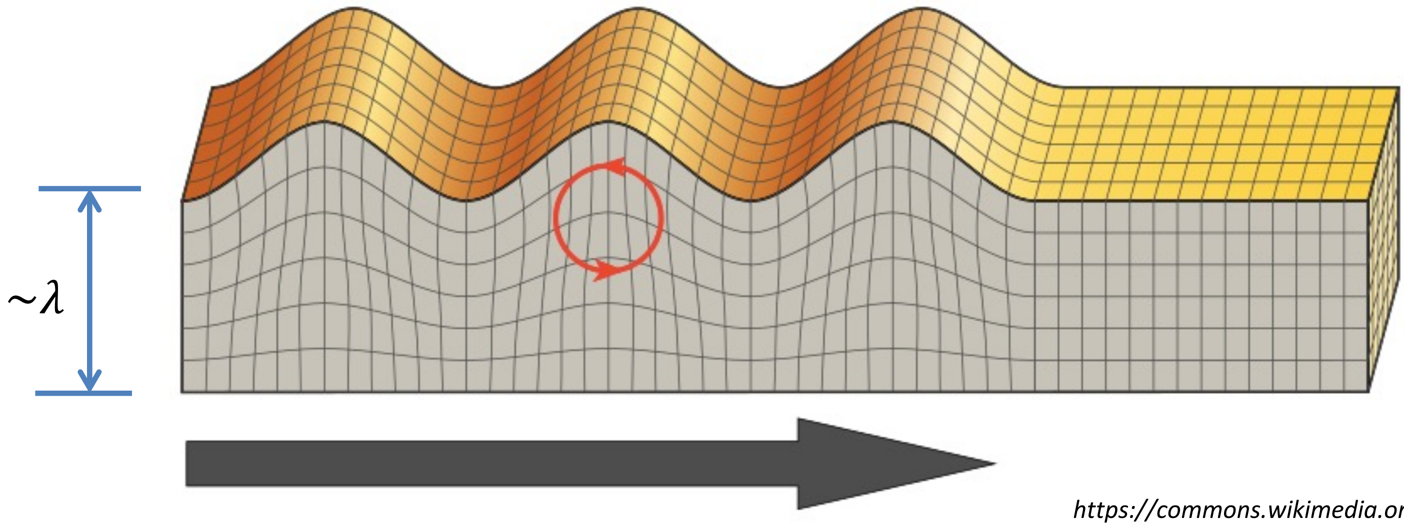
https://en.wikipedia.org/wiki/Seismic_wave#/media/File:Seismic_wave_travel_through_Earth.png



https://www.vox.com/2015/5/13/8595157/inge-lehmann

Rayleigh Waves

In addition to bulk P and S waves, seismic disturbances generate surface waves. The most common is the *Rayleigh wave*, shown below and predicted by Lord Rayleigh in 1885. It's a combination of longitudinal and vertical transverse motion shown by the arrows, rather like waves on the surface of the ocean. The motion is elliptical with counterclockwise particle motion. The displacements from the wave penetrate into the surface a distance of order one wavelength. On the ocean the restoring force is gravity while here it's the elasticity of the solid. Rayleigh waves on the earth's surface travel about 7800 mph (3200 m/sec) and often cause the most disruption from an earthquake.



https://commons.wikimedia.org/wiki/File:Rayleigh_wave.svg

Very low frequency (< 1 Hz) Rayleigh waves are typical of seismic activity. Higher frequency Rayleigh waves (100 kHz) are used For nondestructive testing of structures. Very high frequency ones (10^7 Hz – 10^{10} Hz) are used throughout the electronics industry, particularly for precisely defined radio frequency filters. Within the past 20 years, Rayleigh wave devices have been exploited for chemical and gas sensing and for microfluidics of biological specimens.

Non-destructive testing with Rayleigh waves

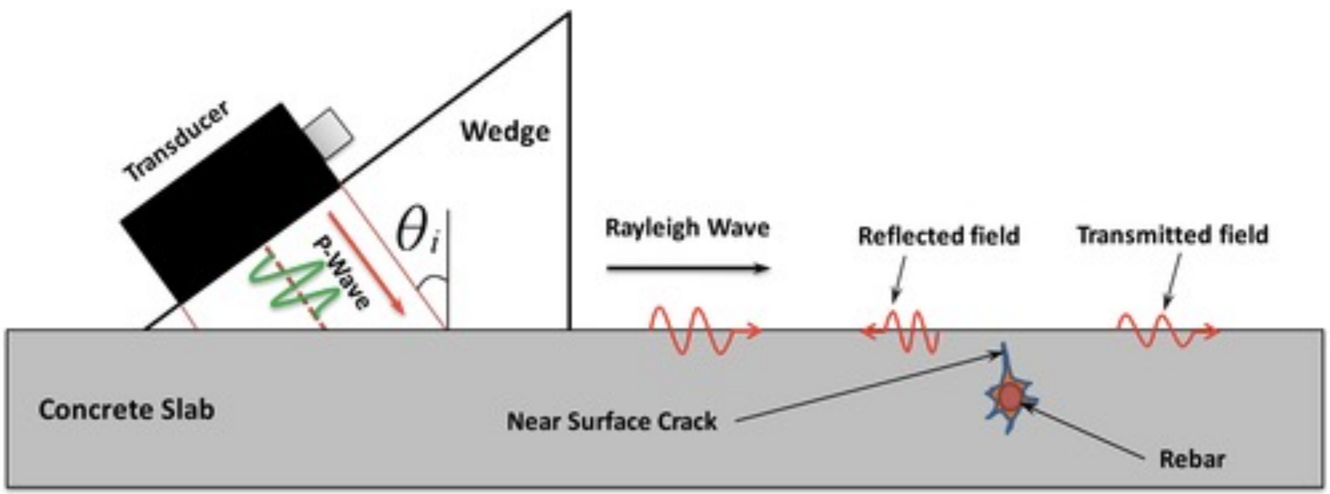
Since they are confined to within about a wavelength of the surface, Rayleigh waves can be particularly useful to problems just beneath the surface. This can occur with reinforced concrete structures in which the steel reinforcing bars may have corroded due to tiny cracks in the concrete. That's apparently what caused the collapse of the bridge in Genoa, Italy, which had supporting cables encased in concrete.

Genoa bridge collapse

The scheme shown below uses Rayleigh waves to search for flaws in the reinforced concrete just below the surface. A transducer generates conventional 100 kHz compressional waves in the wedge, made from a material with lower sound velocity than concrete, in this case teflon. If the angle of incidence satisfies the condition for total internal reflection,

$$\sin\theta_i = \frac{c_{wedge}}{c_{Rayeigh}}$$

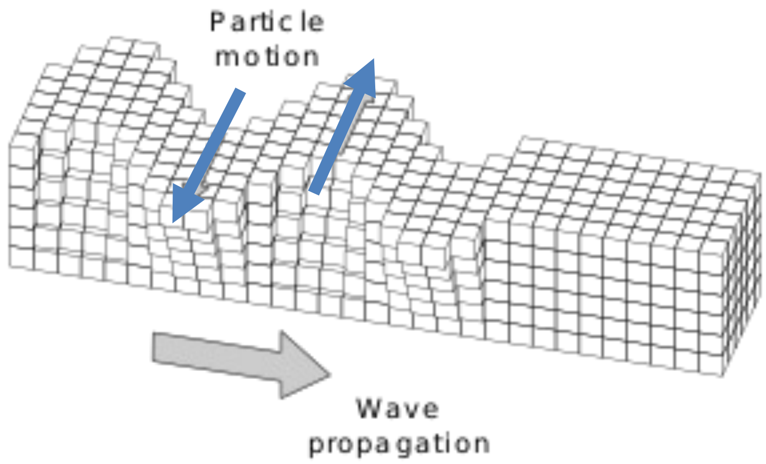
then Rayleigh waves are generated. These then interact with cracks that are within about one wavelength of the surface and reflect some of the sound back, which can be detected with a second transducer.



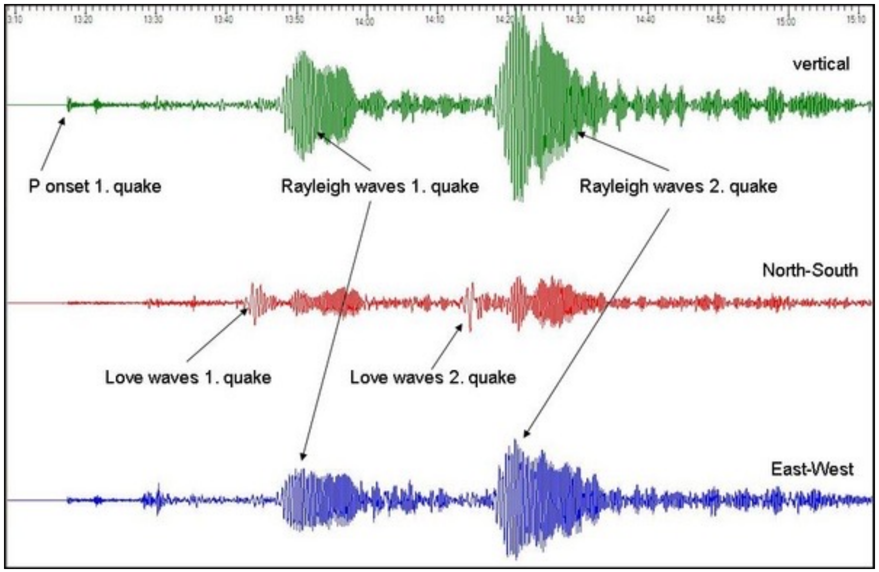
https://onlinelibrary.wiley.com/doi/full/10.1002/stc.2246

Love Waves

Love waves are the second kind of sound wave confined to the surface of a solid. The displacement in a Love wave is *transverse* and *purely horizontal*. Unlike Rayleigh waves, Love waves require a layer on top of the underlying solid in which the displacement is mostly confined. The layer should have a lower sound velocity than the surface. They were first predicted by A.E.H. Love in 1911. For him, the thin layer of interest was the earth's crust, whose thickness (about 15 km on average) he wanted to determine. Love waves from seismic activity travel about 10,000 mph along the earth's surface.



https://en.wikipedia.org/wiki/Love_wave



The plots show signals from two successive earthquakes in Papua New Guinea back in 2010. The direction of propagation was west to east across the Pacific. Seismometers in California were able to distinguish all three components of the surface motion from each of the two quakes. Vertical motion (green) came from Rayleigh waves as did East-West (longitudinal) motion. The North-South ground motion, in red, came from Love waves. The P and S bulk waves generally arrive earlier than Rayleigh and Love waves.

From Berkeley Seismology Lab, *https://seismo.berkeley.edu/blog/2010/07/19/a-race-across-the-pacific-ocean.html*)

Surface acoustic waves (SAW) devices

Electronic devices that make use of Rayleigh waves have been around since the 80's and are integral to all kinds of high frequency equipment like cell phones. They're called SAW filters and the basic layout is shown below. The substrate is a piezoelectric. $LiNbO_3$ is a widely used material but there are many others. On one end a periodic array of metallic fingers is patterned. This is called an interdigital array or IDT. It's driven by a generator at some frequency *f*. The fingers produce a periodic electric field in the piezoelectric which, in turn, launches a Rayleigh wave. The wave travels some distance down the surface and encounters another IDT at the other end. By the inverse piezoelectric effect, the incoming wave generates a time-varying voltage across some load impedance Z_L which might be the input to an amplifier.

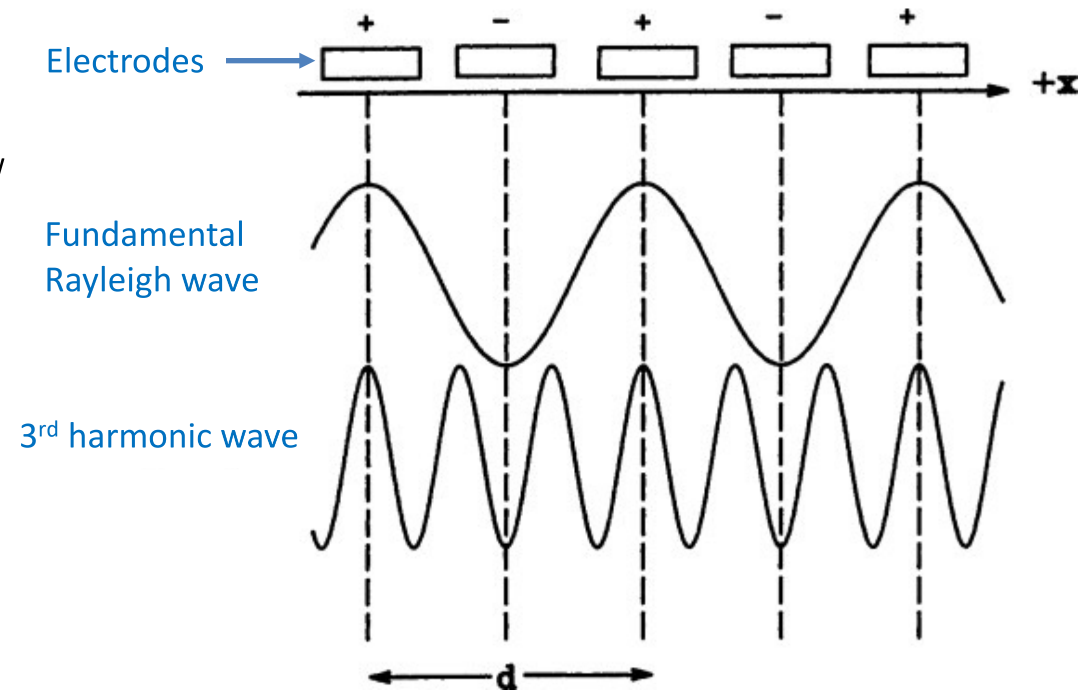
The system is resonant in that a signal whose frequency corresponds to sound whose wavelength equals the period of the digits is preferentially generated and received. Other frequencies are highly suppressed so the device acts as an electrical filter. Instead of inductors and capacitors, which are large, lossy and drift with temperature, the filtering action in a SAW takes place through the interference of Rayleigh waves. SAW filters are cheap, widely available and indispensable for mobile communications. They can be designed to operate anywhere from a few MHz to a few GHz. The upper frequency limit to the SAW is limited by by the spacing *d* between electrodes. For one widely used crystalline cut of $LiNbO_3$, the sound velocity is 3960 m/sec so f = 2 GHz corresponds to d = 2000 nm.

SAW filters

The figure is a side view of the interdigital electrodes patterned on top of the piezoelectric SAW substrate. They are driven periodically with alternating voltages. Resonance is achieved when,

$$d = m\,\lambda_{Rayleigh} = m\frac{c_{Rayleigh}}{f}$$

SAWs are usually driven near the fundamental frequency, *m* = 1.



https://www.sciencedirect.com/topics/physics-and-astronomy/interdigital-transducer

Frequency selectivity

For a device to be an electrical filter we need to know how it responds to all frequencies. The SAW filter operates like a diffraction grating. Label the *N* fingers from left to right *n* = 0, 1, 2,…, *N* -1 … located at x = 0, d/2, d, 3d/2, …. These apply alternating voltages,

$$V_n = (-1)^n\,V\,e^{-i\omega t}$$

The Rayleigh wave amplitude A_n from each electrode is proportional to its exciting voltage V_n. The wave amplitude some position x to the right of the interdigital array, will be the sum of the waves from each electrode. But each wave must travel a distance *d*/2 less than its neighbor to the left so it picks a phase shift -*kd*/2 , just like a diffraction grating. The Rayleigh wave amplitude at (x,t) is the sum,

$$A(x,t) \propto \sum_{n=0}^{N-1} V(-1)^n e^{ik(x-nd/2)}\,e^{-i\omega t} = Ve^{i(kx-\omega t)}\sum_{n=0}^{N-1}\left(-e^{-ikd/2}\right)^n = Ve^{i(kx-\omega t)}\,\frac{1-\left(-e^{-ikd/2}\right)^N}{1-\left(-e^{-ikd/2}\right)}$$

A high frequency filter is usually designed to pass frequencies in a narrow band close to the fundamental frequency defined by,

$$f_1 = \frac{c_{Rayleigh}}{\lambda_1} = \frac{c_{Rayleigh}}{d} \quad \rightarrow \quad k_1 d = 2\pi$$

In our expression for the wave amplitude, write the wavenumber as the fundamental plus the deviation from it,

$$k = k_1 + \Delta k = \frac{2\pi}{d} + \Delta k \quad \rightarrow -e^{-ikd/2} = -e^{-ik_1 d/2} e^{-i\Delta k d/2} = e^{-i\Delta k d/2}$$

Putting this into the expression for the wave amplitude gives,

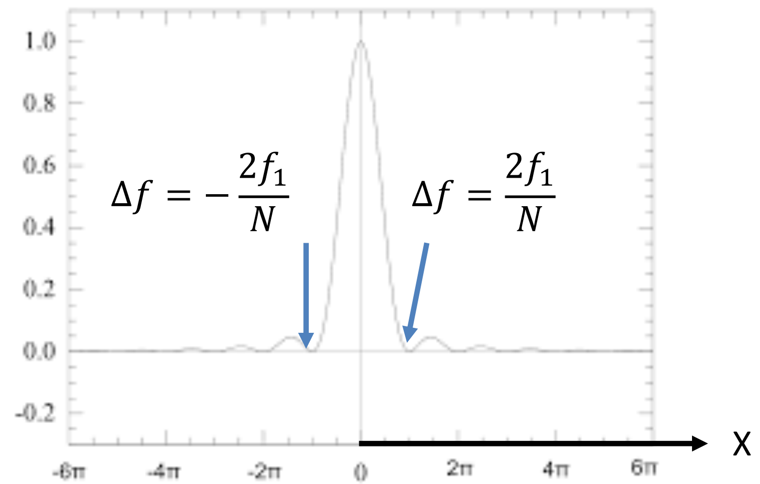
$$A(x,t) \propto V e^{i(kx-\omega t)} \frac{1-\left(-e^{-ikd/2}\right)^N}{1-\left(-e^{-ikd/2}\right)} = V e^{i(kx-\omega t)} \frac{1-\left(e^{-i\Delta kd/2}\right)^N}{1-\left(e^{-i\Delta kd/2}\right)} = V e^{i(kx-\omega t)} \frac{e^{-iN\Delta kd/4}}{e^{-i\Delta kd/4}} \frac{\sin(N\Delta kd/4)}{\sin(\Delta kd/4)}$$

Now convert the deviation in wavenumber to the corresponding deviation in frequency,

$$\Delta k \frac{d}{4} = \frac{\Delta\omega}{c_{Rayleigh}} \frac{d}{4} = \frac{2\pi \Delta f}{c_{Rayleigh}} \frac{d}{4} = \frac{\pi \Delta f}{2 f_1}$$

What we generally care about in filters is the absolute square of the amplitude because that is what the receiving interdigital array will eventually register. Therefore,

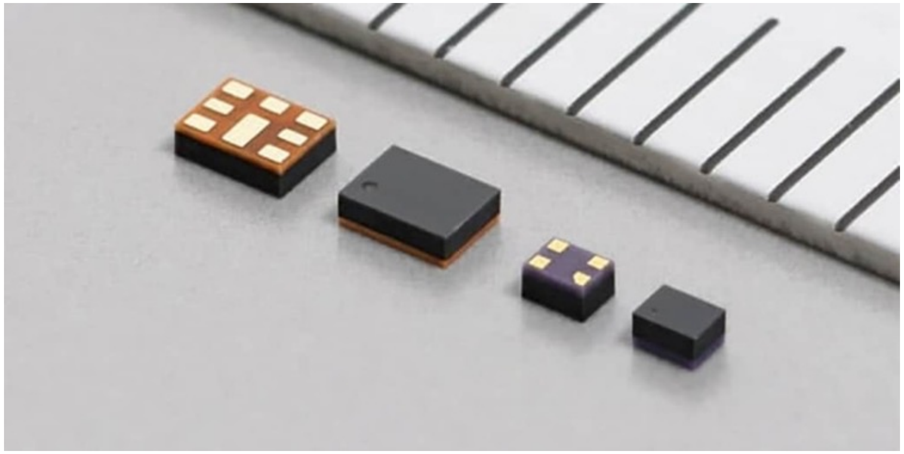
$$Output\ voltage \propto |A|^2 = \left(\frac{\sin N \frac{\pi \Delta f}{2 f_1}}{\sin \frac{\pi \Delta f}{2 f_1}} \right)^2 = \left(\frac{\sin NX}{X} \right)^2$$

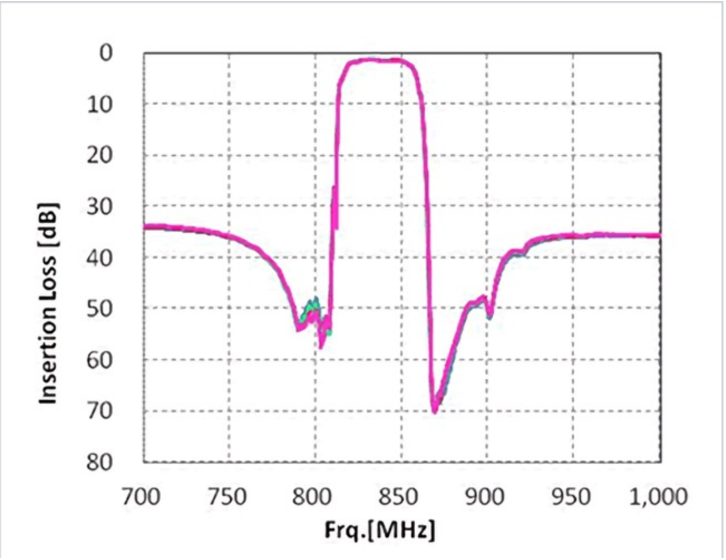


It's the diffraction function, not surprisingly. The plot shows the response as a function of the dimensionless variable X. The bandwidth of the filter is defined by the values of Δf when the output voltage first falls to zero.

$$Bandwidth = \frac{4 f_1}{N} \ll f_1 \qquad N \gg 1$$

By adjusting the pattern of both the transmitting and receiving interdigital arrays it's possible to synthesize all kinds of filter functions. They come encased in packages like the ones shown below – some less than a mm in size. They are used extensively in cell phones to select out specific frequencies and strongly attenuate nearby ones. The filter transfer shown below (right) is designed for a cell phone passband of 820 – 850 MHz. The response falls *very* rapidly once the signal frequency moves out of the passband. SAW filters operate in the frequency range from about 10 MHz - 2000 MHz.

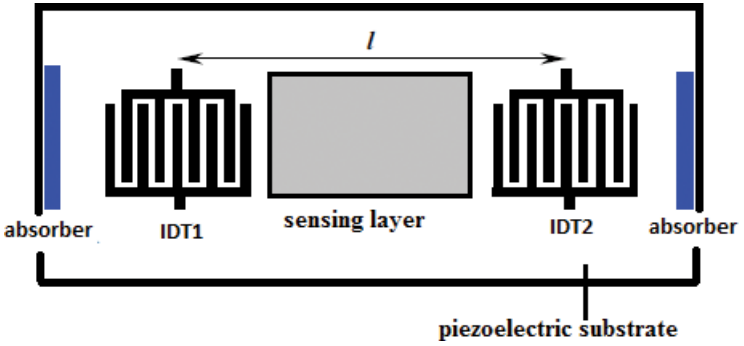


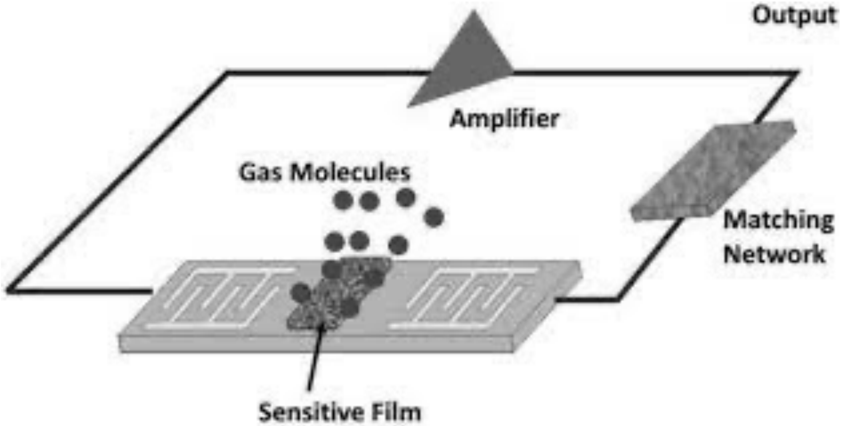


https://www.murata.com/en-global/products/inductor/overview/app/app1/mobile/mobileindex/saw

SAW sensors

SAWs are widely used as sensors. You might want to detect a particular gas like carbon monoxide. Then coat the middle region of the SAW with a film of some chemical that traps or reacts with the CO. This changes the velocity of the surface acoustic wave and therefore it will change the amplitude and phase of the wave reaching the second interdigital array. The extreme sensitivity of the SAW to frequency then leads to big changes in the voltage that is read out of the second array and provides a way to detect small amounts of absorbed gas.

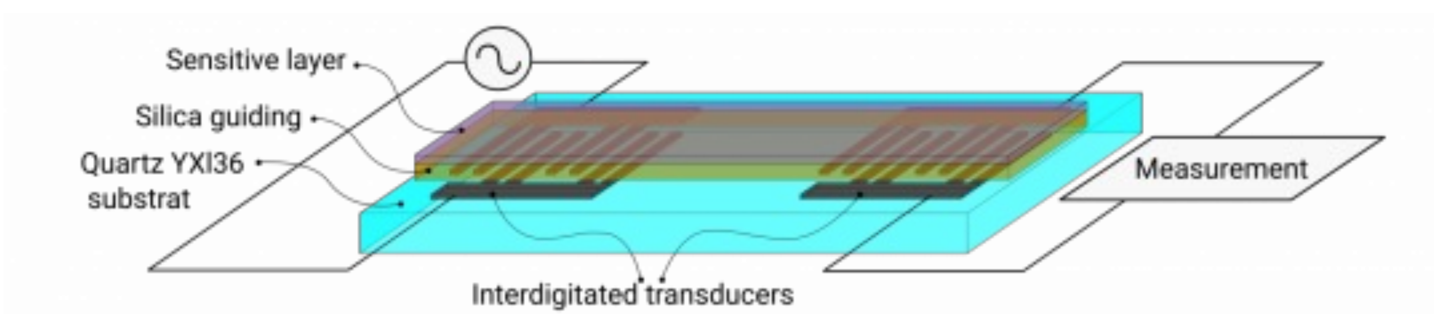




SAW sensors like this are usually operated as oscillators. The amplifier drives the right IDT which sends a sound wave across the sensing region to the left IDT. The signal it receives goes back to the amplifier in a feedback loop. If the gain and phase conditions are adjusted correctly the feedback loop oscillates near the SAW fundamental frequency. The absorption of gas molecules changes the wave velocity and shifts the frequency of oscillation, which can be measured very accurately.

Sensors based on Love waves

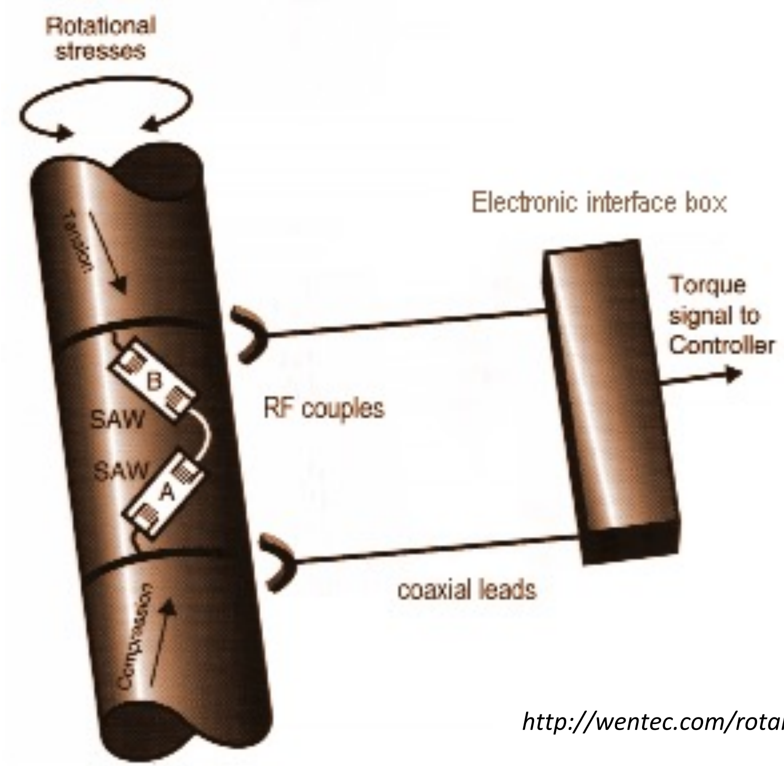
Love waves are are also used in SAW sensors. To detect a particular compound like formaldehyde, the surface of the SAW must be *functionalized*, i.e., coated with a film of something that reacts with the formaldehyde. If the speed of sound in the film is less than in the substrate, Love waves can travel along the SAW transducer, mostly confined to the thin film. In other cases, the thin film may itself be piezoelectric, as in the case of ZnO. Again, if the conditions are right, Love waves will be propagated between the interdigital transducers and their velocity will be slightly changed by the absorption of gas in the sensitive layer.



https://teams.femto-st.fr/equipe-cosyma/surface-acoustic-waves-sensors-environmental-control

Strain sensing

SAWs can be used to detect changes in torque a torque sensor. The piezoelectric substrate of the SAW is mounted directly on a rotating shaft. Rotational stresses in the shaft translate into changes of the SAW resonant frequency, something that, again, can be measured to parts in 10^9. Here, the SAWs are excited and read out by wireless electronics.

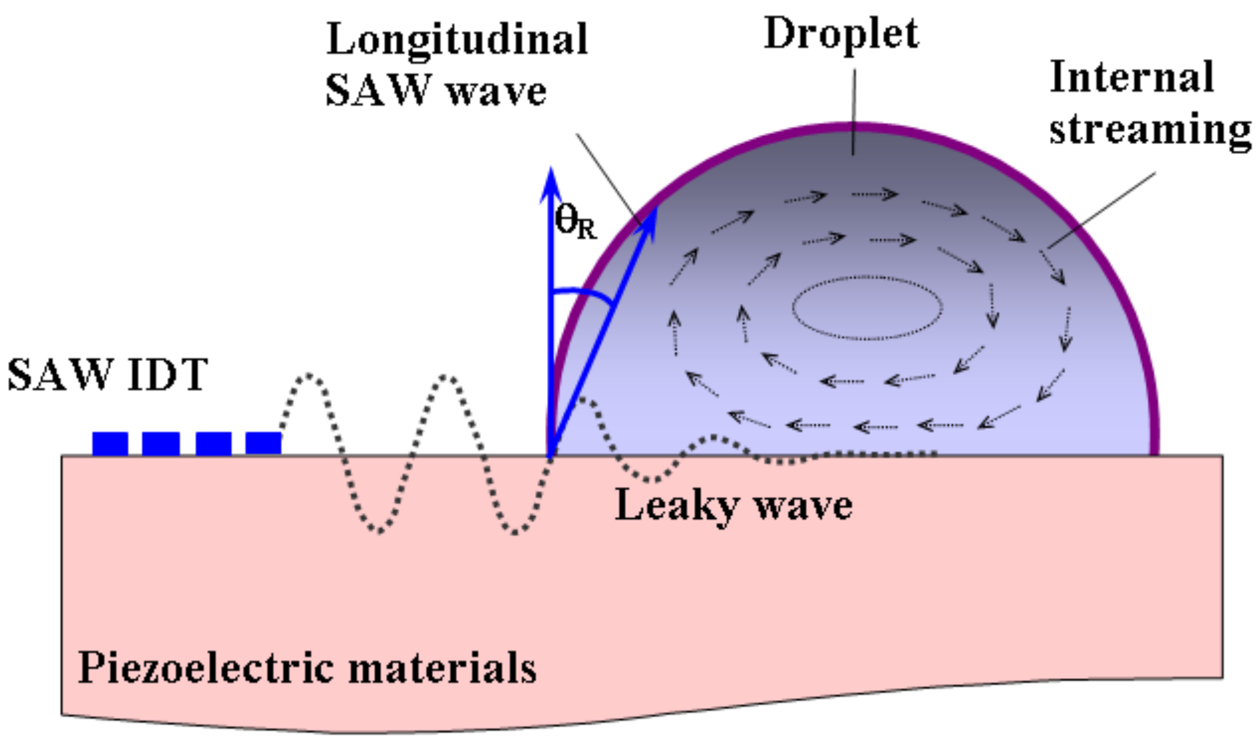


http://wentec.com/rotarytorque/sawprimer/

SAW microfluidics

In the past few decades there has been considerable motivation to do biological analysis faster and cheaper. The ultimate goal is the "lab of a chip" in which micro-liters or less of blood or some other material can be rapidly moved around, stirred, mixed, separated and analyzed. All of this requires doing fluid mechanics on a very small (micron) scale – *microfluidics*. Ordinary fluid flow in and around very small objects is dominated by viscosity (*low Reynolds number* for fluid mechanics experts.) It's diffusive and therefore slow which makes mixing and stirring difficult.

The figure below is an example of where SAW transducers come in. A drop of the material to be studied is deposited on the surface of a SAW device. The IDT generates surface waves which then interact with the droplet. The Rayleigh waves can generate longitudinal sound waves in the droplet that propagate at the angle θ_R . If the sound amplitude is high enough, nonlinear terms in the Navier-Stokes equation will generate fluid flow shown inside the drop. This kind of flow is called *acoustic streaming,* first observed and explained by Lord Rayleigh. Streaming is much more efficient at mixing and stirring than diffusion. It can also provide a way to physically move fluids around on the surface.



From "Acoustic Wave Based Microfluidics and Lab-on-a-Chip", J.K. Luo, Y.Q. Fu, W.I. Milne, DOI: 10.5772/56387

https://www.intechopen.com/chapters/45575

Acoustic streaming

Streaming is a *nonlinear* acoustic effect. A high frequency acoustic disturbance of large enough amplitude will excite nonlinear terms in the Navier Stokes equation. The theory is involved but the basic approach is like perturbation theory. The density and fluid velocity are now written as expansion with successively higher order terms,

$$\rho = \rho_0 + \rho' = \rho_0 + \rho_1' + \rho_2' + \cdots \qquad \vec{v} = \vec{v}_1 + \vec{v}_2 + \cdots$$

ρ_1' and $\vec{v}_1$ are density and velocity in the original sound wave. Focus just on the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = \frac{\partial}{\partial t}(\rho_0 + \rho_1' + \rho_2' + \cdots) + \nabla \cdot \big((\rho_0 + \rho_1' + \rho_2' + \cdots)(\vec{v}_1 + \vec{v}_2 + \cdots)\big) = 0$$

The first order version is linear,

$$\frac{\partial \rho_1'}{\partial t} + \rho_0 \nabla \cdot \vec{v}_1 = 0$$

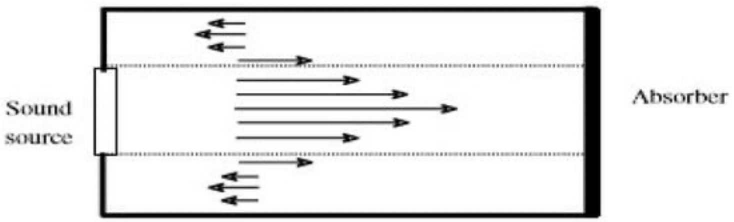
The second order version contains a *product* of the density and velocity of the original sound wave, ρ_1' and $\vec{v}_1$.

$$\frac{\partial \rho_2'}{\partial t} + \rho_0 \nabla \cdot (\vec{v}_2) + \nabla \cdot (\rho_1' \vec{v}_1) = 0$$

You can think of it as a driving force for the higher order terms ρ_2' and $\vec{v}_2$. But if ρ_1' and $\vec{v}_1$ both vary as $\sin \omega t$ then the product will have a term that is *independent of time since,*

$$\sin \omega t \sin \omega t = \frac{1}{2} + \frac{1}{2} \cos 2\omega t$$

The full derivation requires us to do the same expansion with the equation of motion, including viscosity. In the end, you obtain with a time-independent velocity field $\vec{v}_2$ that varies in space but is driven by the acoustic pressure disturbance oscillating at the frequency ω. The figure shows streaming in a closed tube with a sound source at one end. The streaming flow vectors are shown by arrows.



Junru Wu, *Fluids* **2018**, *3*(4), 108; https://doi.org/10.3390/fluids3040108

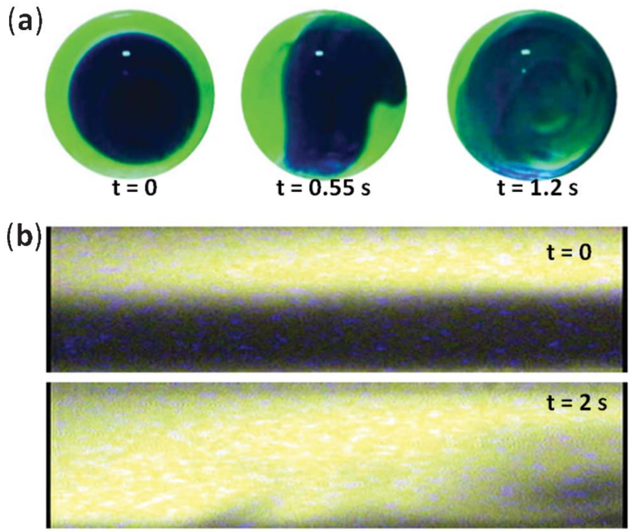
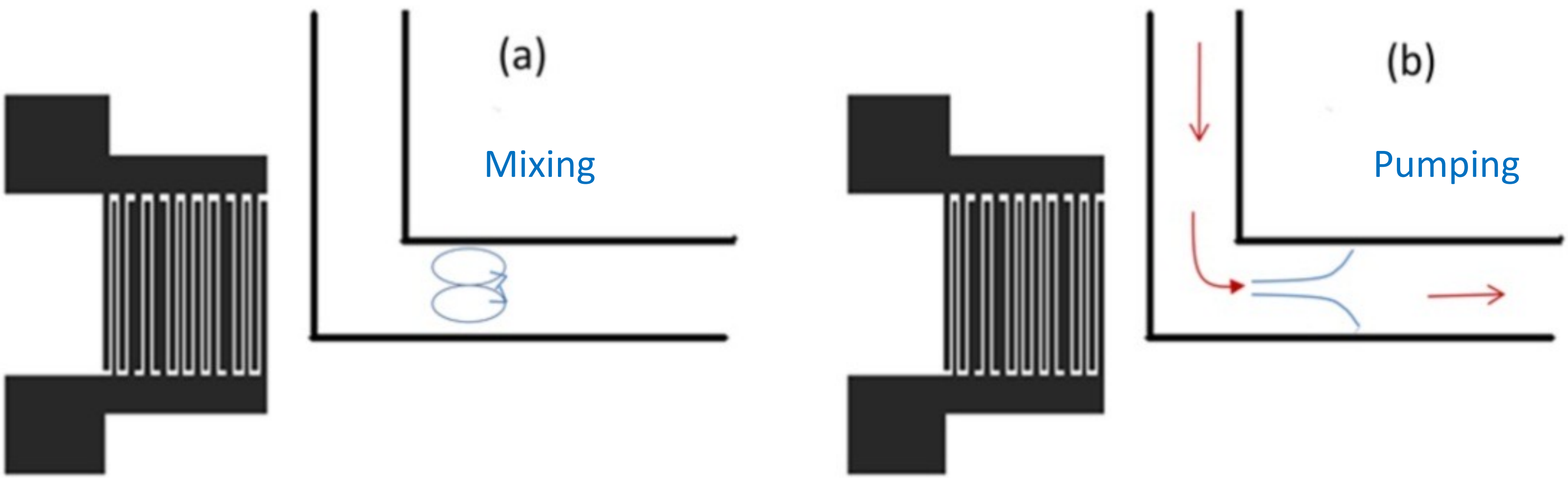


Fig. 3 Fluid mixing by SAW-induced acoustic streaming. (a) Rapid mixing of glycerine (light) and water (dark) in a droplet. (b) Mixing of fluorescence dyes (light) with water (dark) in a rectangular microfluidic channel. Reprinted with permission from ref. 73 and 54.

The left figure shows how SAW-induced acoustic streaming is used to mix glycerine and water in both droplets and microfluidic channels.

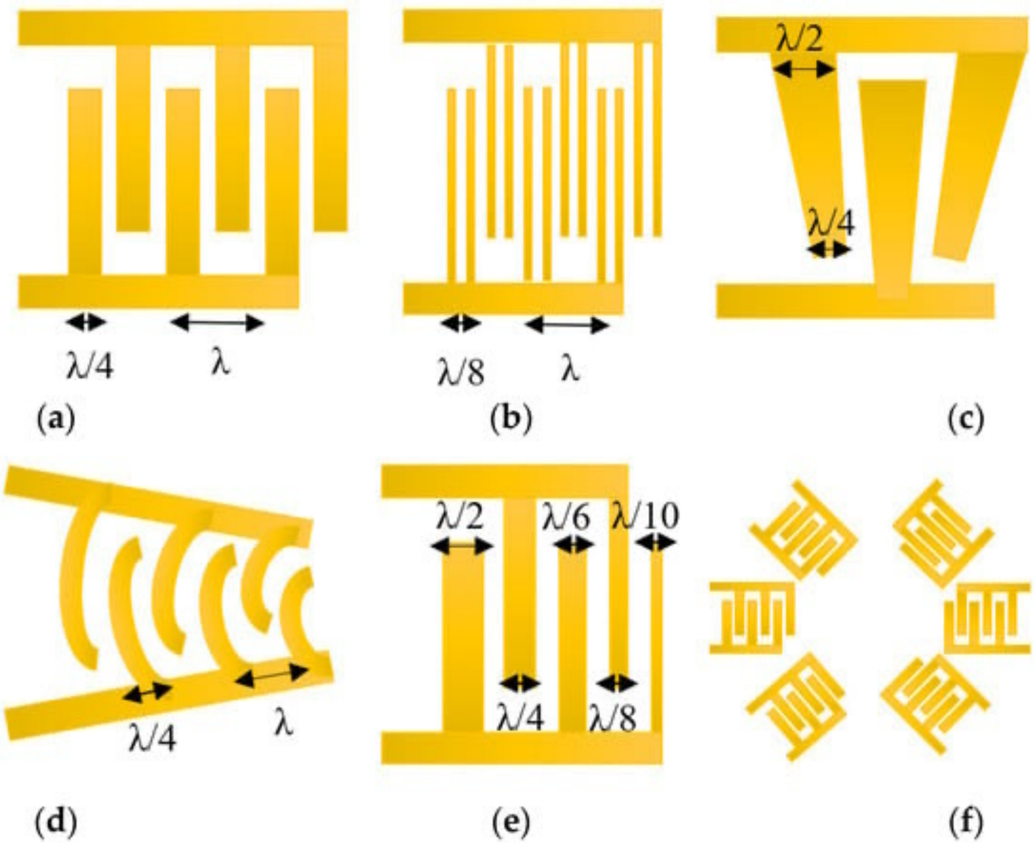
From: Lab Chip ***13****, 3626 (2013) Surface Acoustic Wave Microfluidics, X. Ding et. al.*

SAW-generated streaming tends to produce vortex-like flow patterns within the droplet. These can be used to pump liquid through tiny channels. The interdigital array is on the left and a channel is fabricated onto the substrate. The SAW produces a streaming vortex as shown in (a) and (b). If the vortex is smaller than the width of the channel then things mix but the fluid doesn't move. On the other hand, if the vortex is bigger than the width of the channel there can be net flow of fluid and the device acts like a pump.



From "Acoustic Wave Based Microfluidics and Lab-on-a-Chip", J.K. Luo, Y.Q. Fu, W.I. Milne, DOI: 10.5772/56387

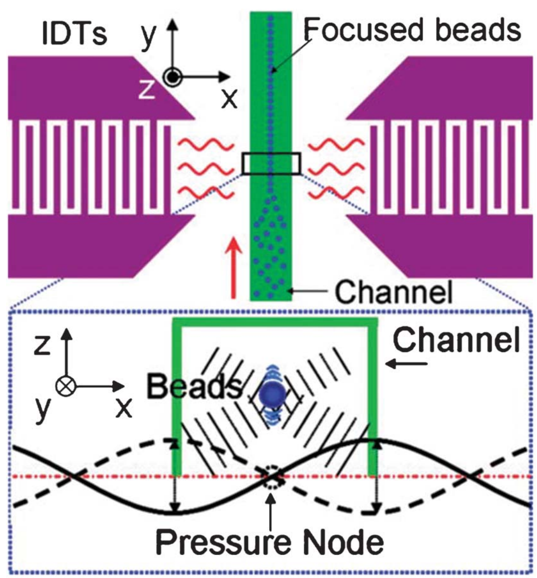
The need to position and move cells around has lead to a lot of imaginative designs beyond the standard interdigital array shown in (a). Since the IDTs can be patterned using standard lithography techniques the designs are liimitless. Pattern (b) sends out waves in just one direction. Pattern (c) can generate more versatile standing wave patterns. Pattern (d) is used to focus waves. Pattern (e) is a *chirp* design that can send out waves over a range of frequencies. Pattern (f) is an array of IDTs to produce custom standing wave patterns in the central region.



https://www.mdpi.com/2072-666X/13/1/30

Acoustic tweezers

Acoustic radiation pressure is a second force that is used to move and sort particles under the influence of SAW acoustic fields. It's proportional $(p')^2$, where p' is the pressure disturbance in the sound wave. Depending on the particle properties (density, compressibility, size) it will force a particle toward either a sound wave pressure node or antinode. The figure below shows the general idea. Opposed IDTs generate a pattern of standing Rayleigh waves which, in turn, produce pressure waves in the fluid of the channel. Particles of interest will be drawn to pressure nodes.



*From: Lab Chip **13**, 3626 (2013) Surface Acoustic Wave Microfluidics, X. Ding et. al.*

The device shown on the right (a) uses 4 IDTs sending out Rayleigh waves to produce a 2-dimensional standing wave pattern in a square chamber. To move the blood cell around it's necessary to move the position of the nodes. That's done by changing the SAW frequency. To do that efficiently, the IDTs must operate over a wide frequency range. That's accomplished with a *chirp* design in which the finger spacing of the IDT varies continuously, similar to a broadband antenna. Panel (b) shows successive locations of the blood cell that form the letters PSU (Penn State University.) Panel (c) shows how the device can move an entire organism (a worm, evidently) from one location to another.

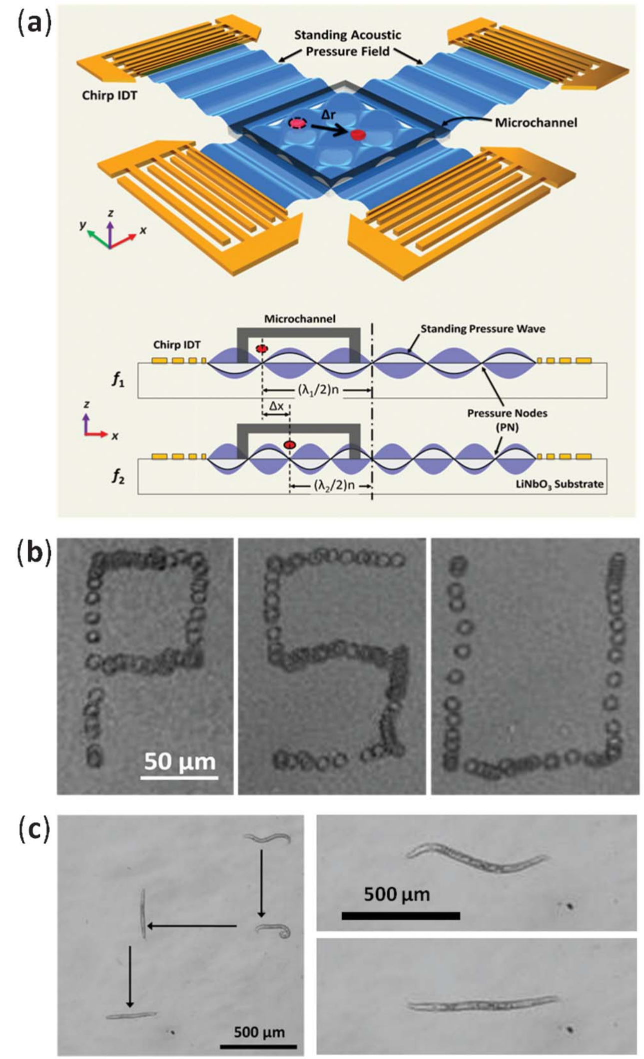


Fig. 21 (a) Device schematic and working mechanism of SSAW-based manipulation of a single particle contained in stagnant fluid (*i.e.*, acoustic tweezers). (b) Composited image of a single bovine red blood cell translated in two dimensions by changing the frequencies of the constituent SAWs. (c) Optical images showing the manipulation and stretching of a whole *C. elegans* worm. Reprinted with permission from ref. 29.

*From: Lab Chip **13**, 3626 (2013) Surface Acoustic Wave Microfluidics, X. Ding et. al.*