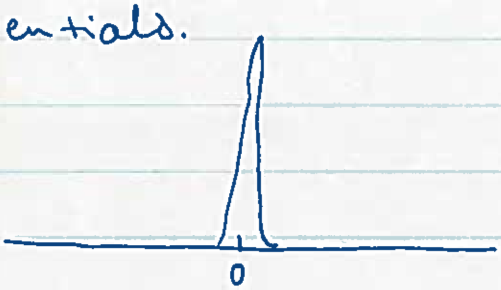
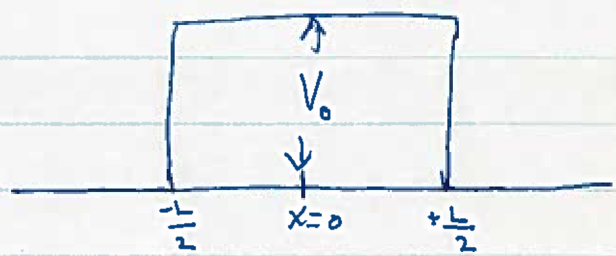
Lecture 9: 9-30-97

1.) 1-d scattering: $V(x) = V(-x)$.

We have been considering the case in which $V(x) \neq V(-x)$. The step potential is just 1 case. We now want to focus on δ-fcn. and square barrier potentials.





Because $\delta(x) = \delta(-x)$, both of these potentials have the property that $V(x) = V(-x)$. In this case $[H, P] = 0 \Rightarrow$ the eigenstates of the parity operator can be used to diagonalize this problem. The two parity states in $d=1$ are $\psi_{k0} = \cos k$, $\psi_{k1} = \sin k$.

$$P\psi_{k0} = \psi_{k0} \qquad \text{even.}$$
$$P\psi_{k1} = -\psi_{k1} \qquad \text{odd.}$$

We will take as our general potential

$$V(x) = 0 \qquad |x| > \frac{L}{2}.$$

We need to take linear combinations of the parity eigenstates $\ni$

$$\psi^+(x) = Te^{ikx} \qquad x > \frac{L}{2}$$
$$= e^{ikx} + Re^{-ikx} \qquad x < -\frac{L}{2}.$$

As in the box problem, we can write the Parity eigenstates as $\cos(k \pm \delta)$ and $\sin(k \pm \delta)$, where δ is a phase shift. Reflection symmetry of the potential can be restated as an invariance under rotation about the $x=0$ axis by $180°$. There are only two directions about this axis that preserve this symmetry: $\longrightarrow$ and $\longleftarrow$.
$\Rightarrow$ there should only be 2 phase shifts for $d=1$. We will call these δ_0 and δ_1. We rewrite the Parity states as

$$\left.\begin{aligned} \psi_0 &= \cos(kx+\delta_0) \\ \psi_1 &= \sin(k+\delta_1) \end{aligned}\right\} \quad x > L/2.$$

$$\left.\begin{aligned} &= \cos(kx-\delta_0) \\ &= \sin(kx-\delta_1) \end{aligned}\right\} \quad x < -\frac{L}{2}.$$

We will see that δ_0 represents the phase shift ~~that~~ for the even states and δ_1 for the odd states.

We now have to take the correct linear combinations such that $\psi^+(x>L/2)$ and $\psi^+(x<-L/2)$ have the form we proposed. For $x > L/2$, we have that

$$\psi^+ = Te^{ikx} \Rightarrow \psi^+ = e^{i\delta_0}\psi_0 + ie^{i\delta_1}\psi_1$$

$$= \frac{1}{2}\left(e^{2i\delta_0} + e^{2i\delta_1}\right)e^{ikx} \quad x > \frac{L}{2}$$

$$\Rightarrow T = \frac{1}{2}\left(e^{2i\delta_0} + e^{2i\delta_1}\right)$$

$$= \frac{1}{2}\left[(e^{2i\delta_0} - 1) + (e^{2i\delta_1} - 1)\right] + 1$$

$$= 1 + i\sum_{l=0,1} e^{i\delta_l}\sin\delta_l .$$

$$\Rightarrow \quad |T|^2 = \cos^2(\delta_0 + \delta_1).$$

Now Let's look at $x < -L/2$.

$$\psi^+ = e^{ik} + Re^{-ikx} \Rightarrow \psi^+ = e^{i\delta_0}\psi_0 + ie^{i\delta_1}\psi_1 \qquad \Big\} x < -L/2$$

$$= \frac{1}{2}(e^{i2\delta_0} - e^{2i\delta_1})e^{-ikx} + e^{ikx} \Big].$$

$$\Rightarrow \quad R = \frac{1}{2}(e^{2i\delta_0} - e^{2i\delta_1})$$

$$\Rightarrow \quad |R|^2 = \sin^2(\delta_0 + \delta_1).$$

$\Rightarrow \quad |R|^2 + |T|^2 = 1.$ This is conservation of probability. Let us do a bit more algebra. Let's add and subtract 1 from R.

$$R = \frac{1}{2}\left[(e^{2i\delta_0} - 1) - (e^{2i\delta_1} - 1)\right].$$

$$= i\sum_{l=0,1} (-1)^l e^{i\delta_l}\sin\delta_l .$$

For the 1-d scattering problem, we see that R and T can be formulated entirely in terms of two phase shifts. The form we have written R and T in reflects the fact that R and T are phase shifted relative to one another. This is why $|R|^2 = \cos^2(\delta_0+\delta_1)$ and $|T|^2 = \cos^2(\delta_0+\delta_1+\pi/2) = \sin^2(\delta_0+\delta_1)$.

As mentioned previously invariance to 180° rotation is the general origin of the two phase shifts. In 3-d the generalization is full rotational invariance under an arbitrary rotation. Such symmetry is called spherical invariance. In 1-d there are only two angles relative to the 180° invariance, 0 and π. In 1-d our "polar" variables are $r = |x|$, $\theta = 0$ if $x > 0$ and $\theta = \pi$, $x < 0$. The associated phase shifts are $\delta_{\theta=0}$ and $\delta_{\theta=\pi}$. In 3-d scattering the incident plane wave is scattered into an outgoing spherical wave of the form $f(\theta)\, e^{ikr}/r$. Let us rewrite our general 1-d solution in terms of r and θ (for the 1-d problem).

$$\left.\begin{matrix} T e^{ikx} & x > L/2 \\ e^{ikx} + R e^{-ikx} & x < -\frac{L}{2} \end{matrix}\right\} \longrightarrow e^{ikx} + g(\theta) e^{ikr} \qquad r > \frac{L}{2}$$

$$g(0) = T - 1$$
$$g(\pi) = R.$$

It is easy to see that this compact form for ψ^+ satisfies our Schroedinger equation. $g(\theta)$ describes entirely the angular dependence of the scattered amplitude. The total scattered intensity is $\sum_\theta |g(\theta)|^2$.

$$\Rightarrow \quad S.I. = |g(0)|^2 + |g(\pi)|^2$$

$$= |R|^2 + |T-1|^2 = 1 - |T|^2 + |T-1|^2$$
$$= 2 - T - T^*$$
$$= 2\,\mathrm{Re}(1-T) = -2\,\mathrm{Re}\, g(0)$$

Note $g(\theta)$ is dimensionless $\Rightarrow$ $|g(\theta)|^2$ can be thought

of as a scattering probability. $\Rightarrow$ the total scattering probability $= -2$ Re of the forward scattering amplitude.
$\Rightarrow$ Once T is known the scattering probability is determined. This of course makes physical sense as it is T that describes transmission. The relation between the scattering probability and $-2\,\mathrm{Re}\, g(0)$ is known as the optical theorem in 1-d. The optical theorem is generally stated for 3-d problems in terms of $\mathrm{Im}\, g(0)$. Let's see how this comes about. The 3-d analogue of ψ^+ is

$$\psi^+(x) = e^{ikx} + f(\theta)\frac{e^{ikr}}{r}$$

where in 3-d, the spherically symmetric scattered wave is e^{ikr}/r. $f(\theta)/r$ must be dimensionless. $\Rightarrow$ $[f(\theta)] = L$. What is the relationship between $f(\theta)$ and $g(\theta)$? We require that

$$\boxed{f(\theta)\frac{e^{ikr}}{r} = g(\theta)\frac{e^{ikr}}{ikr}.}$$ ansatz.

$$\Rightarrow \quad f(\theta) \equiv \frac{g(\theta)}{ik}.$$

$$\Rightarrow \quad \sum_{\theta} |f(\theta)|^2 = -2k^{-2}\,\mathrm{Re}\, g(0) = -2k^{-2}\,\mathrm{Re}[ikf(0)]$$

$$= 2k^{-1}\,\mathrm{Im}\, f(0)$$

$$\Rightarrow \quad \boxed{S.P. = \frac{2}{k}\,\mathrm{Im}\, f(0)}$$ <u>Optical theorem</u>

This is the optical theorem. Note that

$$T = 1 + i \sum_{l=0,1} e^{i\delta_l} \sin\delta_l$$

$$\Rightarrow \quad g(0) = T - 1 = i \sum_{l=0,1} e^{i\delta_l} \sin\delta_l .$$

$$g(\pi) = i \sum_{l=0,1} e^{il\pi} e^{i\delta_l} \sin\delta_l .$$

$$\Rightarrow \quad g(\theta) = i \sum_{l=0,1} e^{il\theta} e^{i\delta_l} \sin\delta_l$$

$$\Rightarrow \quad \boxed{f(\theta) = k^{-1} \sum_{l=0,1} e^{il\theta} e^{i\delta_l} \sin\delta_l}$$

The 3-d generalization of $f(\theta)$ just has $l = 0, 1 \ldots$ ∴ there are an infinite number of phase shifts in higher dimensions. When we really do 3-d scattering, we will show that the invariant is not the parity operator but the angular momentum. Each angular momentum eigenstate has its own phase shift.

2.) δ-fcn. Potential

Let's apply this. We will consider a potential that is very short-ranged. Let k be the wavevector of the scattered wave. The condition on the range of the potential is $k\frac{L}{2} \ll 1$. Note that for any potential of the form $V(x) = 0$ for $|x| > \frac{L}{2}$, the eigen-function spectrum is still unchanged for $E \geq 0$. It still scales as $p^2/2m$. $\Rightarrow$ it is doubly degenerate and continuous. $\Rightarrow$ We can

Write the Energy eigenvalue equation as

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi = E\psi = \frac{\hbar^2 k^2}{2m}\psi.$$

$$\Rightarrow \left(\frac{d^2}{dx^2} + k^2\right)\psi = \frac{2mV\psi}{\hbar^2} = U\psi.$$

$$\boxed{U = \frac{2m\,V(x)}{\hbar^2}}.$$

Now Let's integrate over the range which the potential acts.

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \left(\frac{d^2}{dx^2} + k^2\right)\psi = \int_{-L/2}^{L/2} U\psi.$$

$$\Rightarrow \underbrace{\frac{d\psi(L/2)}{dx} - \frac{d\psi(-L/2)}{dx}}_{\text{change in } \psi' \text{ across the potential.}} + k^2\int_{-L/2}^{L/2}\psi = \int_{-L/2}^{L/2} U\psi.$$

Let's now specialize to a δ-fcn. potential.
Consider now the even-parity solution, $\psi_0 = \cos(kx + \delta_0)$ $x > \frac{L}{2}$, $\cos(kx - \delta_0)$ $x < -L/2$. For a δ-fcn. we need to take the $\lim_{L\to 0}$.

$$k^2 \int_{-0}^{+0} \psi_0 = k^2\left[\psi_0(0) - \psi_0(-0)\right] = 0.$$

set $U(x) = -U_0\,\delta(x)$ (only attractive case)

$$\Rightarrow \quad \psi_0'(+0) - \psi_0'(-0) = -U_0 \int_{-0}^{0} \delta(x)\, \psi_0$$

$$\Rightarrow \quad -2k \sin\delta_0 = -U_0 \psi(0) = -U_0 \cos\delta_0$$

$$\Rightarrow \quad \boxed{\tan\delta_0 = U_0/2k.}$$

Note $\delta_0 > 0$ but a bound state forms!!

This is the fundamental relationship that determines the phase shift for the even-parity state. What about the odd-parity state: $\psi_1(0)=0 \Rightarrow \psi_1'(0) - \psi_1'(-0) = 0 \Rightarrow$ the only non-zero term is $k^2[\psi_1(0) - \psi_1(-0)] = k^2[\sin\delta_1 - \sin -\delta_1] = 0$ $\Rightarrow \delta_1 = 0$ or some integral multiple of π. Let's choose $\delta_1 = 0$. $\Rightarrow$ for a δ-fcn potential only the even phase shift survived. For a $+\delta$-fcn., $\tan\delta_0 = -U_0/2k$.

Let's show another relationship. From highschool trig. we know that

$$\sin 2z = \frac{2\tan z}{1+\tan^2 z}$$

$$\Rightarrow \quad \cos 2z = \frac{1-\tan^2 z}{1+\tan^2 z}.$$

$$\Rightarrow \quad \boxed{e^{2i\delta_0} = \frac{1+i\tan\delta_0}{1-i\tan\delta_0} = \frac{2k+iU_0}{2k-iU_0}}$$

Because $\delta_1 = 0 \Rightarrow e^{2i\delta_1} = 1$.

Let us now construct the scattering amplitudes.

$$g(\theta) = \sum e^{i\ell\theta} e^{i\delta_\ell} \sin\delta_\ell$$

$$= i e^{i\delta_0} \sin\delta_0 = \frac{1}{2}\left[e^{2i\delta_0} - 1\right]$$

$$= \frac{iU_0}{2\kappa - iU_0}$$

$\Rightarrow$ $g(\theta)$ is independent of θ. Let's compute $|T|^2$.

$$|T|^2 = |g(\theta)-1|^2 = \frac{(2\kappa)^2}{(2\kappa)^2 + U_0^2}.$$

$\Rightarrow$ $|T|^2 \propto \kappa^2 \propto E_{inc.}$ also $|R|^2 = U_0^2/((2\kappa)^2 + U_0^2)$
$\Rightarrow$ $|R|^2 \propto U_0^2 \Rightarrow |R|^2 = 0$ only if $U_0 = 0$. $\Rightarrow$ a δ-fcn. has no resonances. The independence of $g(\theta)$ on θ signifies that the backward and forward scattering amplitudes are equal. They can only differ when δ_1 and δ_0 are simultaneously $\neq 0$. When $\delta_1 = 0$ $g(0) = g(\pi) \Rightarrow$ $\boxed{T - 1 = R}$.

Let us now construct the wave functions. Note when $\kappa = iU_0/2$, $g(\theta)$ is divergent. $\Rightarrow \lim_{\kappa \to iU_0/2} g(\theta) \to \infty$.
Our wave function is of the form
$\psi^+ = e^{i\kappa x} + g(\theta) e^{i\kappa r}$ $r > L/2(0)$. $\Rightarrow$ we need to renormalize ψ^+ so that the outgoing part remains finite. If $\kappa = iU_0/2$ $i\kappa x \to -U_0/2\, x$ $\Rightarrow$ $x > 0$ for ψ^+ to converge. $\Rightarrow$ there is no incoming wave, only an out going wave of the form

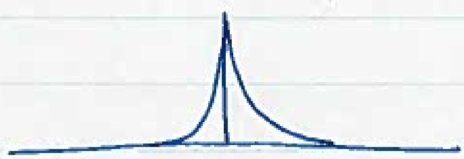
$$\boxed{\psi_{out} = e^{-\frac{U_0}{2} r}}$$

$\boxed{\text{Note } U(x) = -U_0\delta(x)}$

This state forms for $\kappa^2 = -U_0/2 < 0 \Rightarrow$ ψ_{out} describes a bound state. The energy of this state is

$$-\frac{\hbar^2}{2m}\frac{d^2}{dr^2}\psi_{out} = -\frac{U_0^2\hbar^2}{8m}\psi_{out}.$$

Note ψ_{out} is peaked at $r=0$ and decays from there

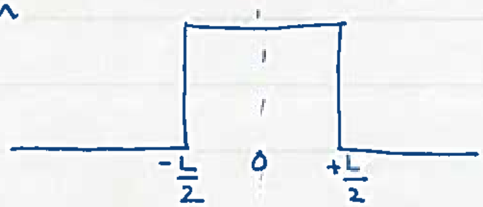


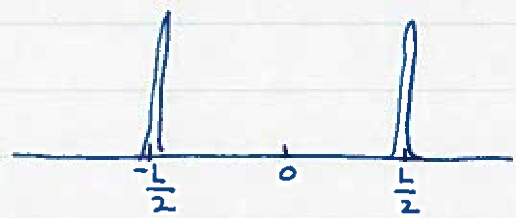
$\Rightarrow$ the entire amplitude of ψ_{out} is contained in a region in which $U_0 = 0$. $\Rightarrow$ the particle spends most of its time in a region where no force acts on it. This is true only for potentials for which $k\frac{L}{2} \ll 1$.
The vanishing of the incoming wave just says that it is impossible to send a particle into the scattering center with negative energy.

Lecture 10: 10-2-97

1.) Resonances:

Today we want to study potentials of the form





Baym works out the square well as do most books. We will consider the 2-δ-fcn. potential as this potential nicely illustrates the principle of resonant scattering. Resonance occurs when $|R|^2 = 0 \Rightarrow$ there is no reflection. A single δ-fcn will always yield reflection. However, with two, the reflection from the second wall can be $180°$ out of phase from the reflection from the first wall. When this occurs, unit transmission occurs. We now want to formulate this circumstance.

Our potential is of the form

$$U(x) = \frac{2m}{\hbar^2} V(x) = -\frac{1}{2} U_0 \left[\delta\left(x + \frac{L}{2}\right) + \delta(x - L/2) \right]$$

When $L=0$ $U(x) \rightarrow -U_0 \delta(x)$, the single δ-fcn. potential. This is a check on our results. The symmetry inherent in this potential guarantees that we can utilize the parity solutions we have written down before. With the single δ-fcn problem, we matched the boundary conditions around $0^{\pm}$. We now have to match the boundary conditions around $\pm\frac{L}{2} \pm \varepsilon$ in the limit that $\varepsilon \rightarrow 0$. We only need to consider either $\frac{L}{2} \pm \varepsilon$ or $-\frac{L}{2} \pm \varepsilon$ as a result

of the Symmetry around 0. Let's choose $\frac{L}{2} \pm \varepsilon$.
For $L/2 - \epsilon$, we use the free particle parity states and for $L/2 + \varepsilon$, we use the parity scattering states. We integrate the S.E. between $L/2 - \epsilon$ to $L/2 + \epsilon$ and obtain

$$\frac{d}{dx}\psi\left(\frac{L}{2}+\varepsilon\right) - \frac{d\psi}{dx}\left(\frac{L}{2}-\varepsilon\right) + \kappa^2 \int_{\frac{L}{2}-\varepsilon}^{L/2+\varepsilon} \psi\, dx = \int_{\frac{L}{2}-\epsilon}^{L/2+\varepsilon} U\psi$$

$$= -\tfrac{1}{2}U_0\,\psi\left(\tfrac{L}{2}\right).$$

In the limit that $\varepsilon \to 0$ the integral on the LHS $\to \psi(L/2+\varepsilon) - \psi(L/2-\varepsilon)$ which when expanded in a Taylor Series is proportional to ϵ
$\Rightarrow \psi(L/2+\varepsilon) - \psi(L/2-\varepsilon) \to 0$. We are left with

$$\frac{1}{\psi(\frac{L}{2})}\frac{d}{dx}\psi\left(\tfrac{L}{2}+\varepsilon\right) - \frac{1}{\psi(\frac{L}{2})}\frac{d}{dx}\psi\left(\tfrac{L}{2}-\varepsilon\right) = -\tfrac{1}{2}U_0.$$

Note $\psi(\frac{L}{2})$ has to be continuous across the boundary $\Rightarrow$ the correct value of $\psi(L/2)$ is chosen to be consistent with $\frac{L}{2} \pm \varepsilon$.

<u>Case I</u>: $x > \frac{L}{2}$, | $x < -\frac{L}{2}$

	$x > \frac{L}{2}$	$x < -\frac{L}{2}$
<u>even</u>:	$\cos(\kappa x+\delta_0) \Rightarrow \psi_0^{-1}\frac{d\psi_0}{dx} = -\kappa\tan(\kappa x+\delta_0)$	$\cos(\kappa x-\delta_0) \Rightarrow \psi_0^{-1}\frac{d\psi_0}{dx} = -\kappa\tan(\kappa x-\delta_0)$.
<u>odd</u>:	$\sin(\kappa x+\delta_1) \Rightarrow \psi_1^{-1}\frac{d\psi_1}{dx} = \kappa\cot(\kappa x+\delta_1)$	$\sin(\kappa x-\delta_1) \Rightarrow \psi_1^{-1}\frac{d\psi_1}{dx} = \kappa\cot(\kappa x-\delta_1)$

Case II: $-\frac{L}{2} \leq x \leq \frac{L}{2}$

In between the δ-fcn's, the particle is free. $\Rightarrow$ we can use $\cos kx$ and $\sin kx$.

$$\psi_0 = \cos kx \quad \Rightarrow \quad \psi_0^{-1} \frac{d}{dx} \psi_0 = -k \tan kx.$$

$$\psi_1 = \sin kx \quad \Rightarrow \quad \psi_1^{-1} \frac{d}{dx} \psi_1 = k \cot kx.$$

Now let's put this all together in our continuity equation.

even $\quad -k\tan(k\frac{L}{2} + \delta_0) + k \tan k\frac{L}{2} = -\frac{1}{2} U_0.$

odd $\quad k\cot(k\frac{L}{2} + \delta_1) - k \cot k\frac{L}{2} = -\frac{1}{2} U_0.$

$\Rightarrow$ Our scattering problem is now completely solved because δ_0 and δ_1 can be solved for. The utility of the phase-shift approach should now be evident. It permits a straightforward way of solving difficult scattering problems. Let's simplify these eqs. by using trig. identities.

$$\tan(z_1) \pm \tan z_2 = \frac{\sin(z_1 \pm z_2)}{\cos z_1 \cos z_2}$$

and

$$\cot z_1 \pm \cot z_2 = \frac{\sin(z_2 \pm z_1)}{\sin z_1 \sin z_2}.$$

$$\Rightarrow \quad -\frac{U_0}{2k} = \frac{-\sin\delta_0}{\cos\frac{kL}{2}\cos(\delta_0+\frac{kL}{2})}$$

$$\Rightarrow \quad \frac{2k}{U_0} = \frac{\cos\frac{kL}{2}\left[\cos\delta_0\cos\frac{kL}{2} - \sin\delta_0\sin\frac{kL}{2}\right]}{\sin\delta_0}$$

$$= \cos^2 kL/2 \cot\delta_0 - \cos kL/2 \sin kL/2$$

$$\Rightarrow \quad \cot\delta_0 = \frac{2k}{U_0}\sec^2\frac{kL}{2} + \tan kL/2$$

$$= \frac{2k/U_0 + \sin kL/2 \cos kL/2}{\cos^2\frac{kL}{2}}$$

$$= \frac{4k/U_0 + 2\sin\frac{kL}{2}\cos\frac{kL}{2}}{1+\cos kL} = \boxed{\frac{4k/U_0 + \sin kL}{1+\cos kL}}$$

$\Rightarrow$ we have isolated δ_0. Similar manipulations can be performed to isolate δ_1: we find that

$$\cot\delta_1 = -\cot\frac{kL}{2} + 2k/U_0 \csc^2 kL/2$$

$$= \frac{4k/U_0 - \sin kL}{1-\cos kL} \neq 0 \quad \text{as in the single } \delta\text{-fcn. problem.}$$

Let us investigate the features of the phase shifts.

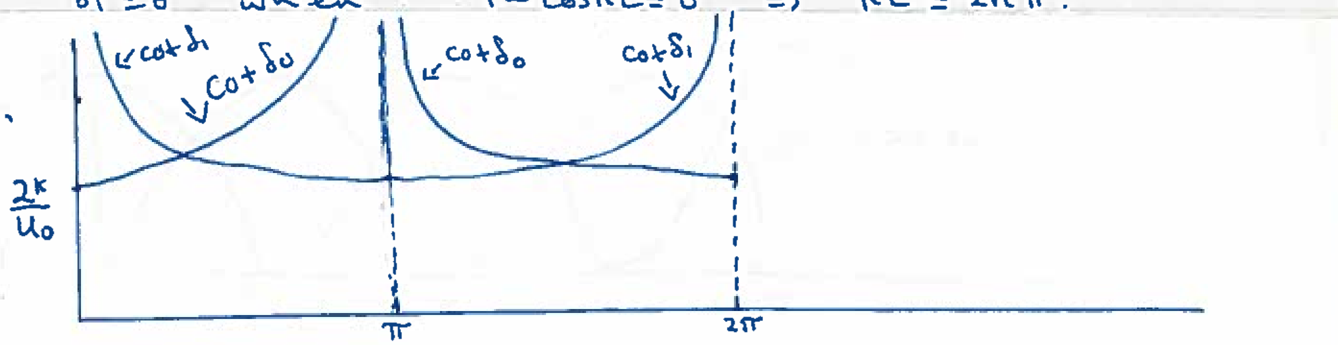
a) $L=0$ $\qquad \tan\delta_0 = U_0/2k$

$$\cot\delta_1 = \infty = 1/0 \Rightarrow \delta_1 = 0$$

This is the single δ-fcn. result. This is true anytime $\sin kL = 0$ and $\cos kL = 1$ $\Rightarrow$ $kL = 2n\pi$ $n = 0, 1, 2, \dots$ the two potentials act as a single δ-fcn.

b.) $\delta_0 = 0$ when $1 + \cos kL = 0$ $\Rightarrow$ $kL = (2n+1)\pi$.

$\delta_1 = 0$ when $1 - \cos kL = 0$ $\Rightarrow$ $kL = 2n\pi$.



$\Rightarrow$ δ_0 and δ_1 are $180°$ out of phase. This is because δ_0 represents forward and δ_1 is the backward scattering. So this fits our intuition.

c.) $k \to 0$

$$\tan\delta_0 \to \frac{2}{4k/U_0 + kL} = \frac{U_0}{2k[1 + U_0L/2]}$$

$$\tan\delta_1 \to \frac{(kL)^2/2}{4k/U_0 - kL} = \frac{kL^2 U_0}{8[1 - LU_0/2]}$$

$\Rightarrow$ $\delta_1 \to 0$ (the back-scattering decreases as $k \to 0$ and δ_0 has a renormalized single δ-fcn. form.

c'.) $k \to \infty$

$$\tan\delta_0 \to \frac{1 + \cos kL}{4k/U_0} \; ; \quad \tan\delta_1 \to \frac{U_0}{4k}(1 - \cos kL)$$

$\Rightarrow$ both oscillate between $U_0/2k$ and 0 for large momentum.

e.) Resonances:

The scattering amplitude is given by

$$g(\theta) = i \sum_{l=0,1} e^{il\theta} e^{i\delta_l} \sin\delta_l$$

$$= \frac{i}{2i} \sum_{l=0,1} e^{il\theta} \left[e^{2i\delta_l} - 1\right] = \frac{1}{2} \sum_{l=0,1} e^{il\theta} \left[e^{2i\delta_l} - 1\right].$$

recall $e^{2i\delta_l} = \dfrac{1 + i\tan\delta_l}{1 - i\tan\delta_l}$.

$$\Rightarrow \boxed{g(\theta) = i \sum_{l=0,1} \frac{e^{il\theta}}{\cot\delta_l - i}}$$ key result.

Recall $g(\pi) = R$.

$$\Rightarrow |R|^2 = \left| \frac{1}{\cot\delta_0 - i} - \frac{1}{\cot\delta_1 - i} \right|^2.$$

$$= \frac{(\cot\delta_1 - \cot\delta_0)^2}{(\cot^2\delta_0 + 1)(\cot^2\delta_1 + 1)} = \sin^2\delta_0 \sin^2\delta_1 (\cot\delta_1 - \cot\delta_0)^2$$

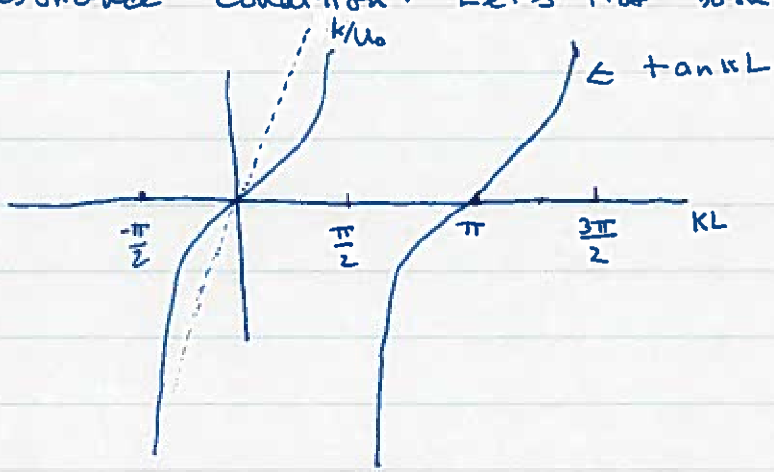
$\Rightarrow |R|^2 = 0$ when $\cot\delta_1 = \cot\delta_0 \Rightarrow$

$$\frac{4k/U_0 + \sin kL}{1 + \cos kL} = \frac{4k/U_0 - \sin kL}{1 - \cos kL}$$

$$\Rightarrow (1-\cos kL)\left(\frac{4k}{U_0} + \sin kL\right) - \left(\frac{4k}{U_0} - \sin kL\right)(1+\cos kL) = 0.$$

$$\Rightarrow \sin kL - \frac{4k}{U_0}\cos kL = 0 \Rightarrow \boxed{\tan kL = \frac{4k}{U_0}}$$

This is the resonance condition. Let's plot both sides.



depending on the magnitude of $4U_0^{-1}$ there may or may not be a solution. As $U_0 \to \infty$ $4/U_0 \to 0$ and no solution will exist $\Rightarrow$ $\exists$ a critical value of U_0, below which a resonance occurs. For small k, $\tan kL \sim kL$. There will be a solution as long as $\frac{4k}{U_0} > kL \Rightarrow \boxed{U_c = 4/L}$.

Note another solution is simply $\sin\delta_0 = 0$ or $\sin\delta_1 = 0$. Either of these conditions is satisfied when δ_0 or $\delta_1 = 0, n\pi, n = 1, 2, \cdots$. δ_0 or $\delta_1 = 0$ is excluded because $\cot\delta_{0(1)} \to \infty$. It is easy to verify that $U_0 = 0$ is the solution to the continuity equation. Consider now δ_0 or $\delta_1 = \pi$.

$$\Rightarrow -\frac{1}{2}U_0 = -k\tan\left(\frac{kL}{2} + \pi\right) + k\tan\frac{kL}{2}$$
$$= 0 \text{ as well.}$$

$$\Rightarrow \boxed{\tan kL = \frac{4k}{U_0}}$$ is the true resonance condition.

The last thing to say about resonances is that the incoming wave at resonance is e^{ikx} because $R=0$.
$\Rightarrow$ ψ_{inc} and ψ_{out} simply differ by at most a phase factor: $\psi_{out} = Te^{ikx}$. In this case $T=1$. $\Rightarrow$ $\psi_{in} = \psi_{out}$. There is no phase shift at resonance for two δ-fcn's.

f.) <u>Nearly</u> <u>Bound</u> <u>States</u>:

Nearly bound states (NBS) are states that trap the particle for a while but eventually decay away. The energy of NBS must be of the form $E - i\alpha \Rightarrow \psi(t) \to e^{-iEt} e^{-\frac{\alpha t}{\hbar}} \psi(t=0)$
$\Rightarrow$ the state decays exponentially with a rate $\alpha/\hbar$. An imaginary part to an energy (as you proved in the 2nd. problem set) represents the decay rate of a state. To locate such states we need to write the scattering amplitude as a function of energy. We note that because $|R|^2 \propto \sin^2\delta_0 \sin^2\delta_1$, $|R|^2$ has maxima when $\sin\delta_\ell = 1 \Rightarrow \cot\delta_\ell = 0$.
Let's expand $\cot\delta_\ell$ as a power series about $\cot\delta_\ell = 0$. Let $E_0 = \hbar^2\lambda^2/2m$ be the energy at which $\cot\delta_\ell = 0$. In the vicinity of E_0

$$\Rightarrow \cot\delta_\ell = \frac{d}{dE}(\cot\delta_\ell)(E-E_0) = \frac{2}{\Gamma}(E-E_0).$$

$$\Rightarrow g(\theta) = i\sum_{\ell=0,1} \frac{e^{i\ell\theta}}{\frac{2}{\Gamma}(E-E_0)-i} = \sum_{\ell=0,1} \frac{e^{i\ell\theta}(\frac{\Gamma}{2})}{E-E_0-i\Gamma/2}.$$

$\Rightarrow$ in the vicinity of E_0, there lies a state with an energy $E = E_0 + i\Gamma/2$. $\Rightarrow$ this state is nearly bound. The particle rattles around and escapes with a rate $\Gamma/\hbar$ or lifetime $\tau = \hbar/\Gamma$. Note as $\Gamma \to 0$ $\tau \to \infty$ $\Rightarrow$ there is no escape from the state with energy E_0.

g.) <u>Bound States:</u>

True bound states are characterized by $\Gamma = 0 \Rightarrow$ there is no escape $\Leftrightarrow$ no imaginary part to the energy. These states are located by the singularities of $g(\theta)$; that is, $\cot\delta_\ell = i$. This can only be true if δ_ℓ is complex. Let's see how this comes about.

$$\cot\delta_\ell - i = \frac{4k/U_0 \pm \sin kL}{1 \pm \cos kL} - i$$

$$= \frac{4k/U_0 \pm \sin kL - i(1 \pm \cos kL)}{1 \pm \cos kL}$$

$$= \frac{4k/U_0 \mp i e^{ikL} - i}{1 \pm \cos kL}$$

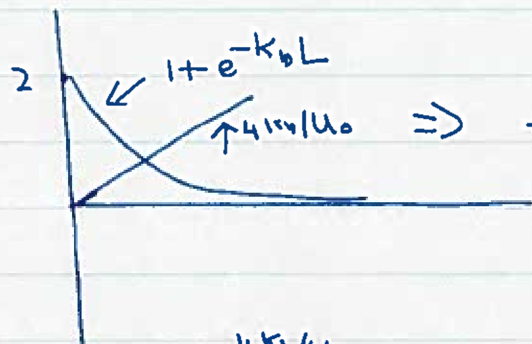
$\Rightarrow$ The bound state condition is

$$\frac{4k}{U_0} = i(1 \pm e^{ikL})$$

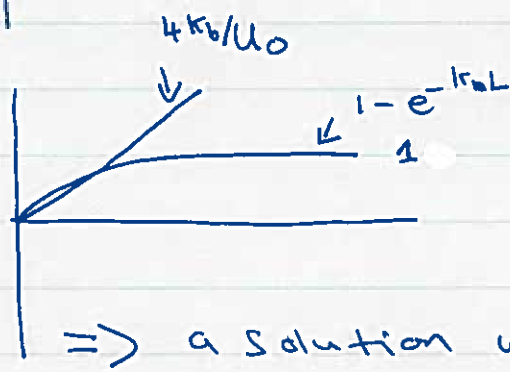
$\Rightarrow$ k must be imaginary: $k = \sqrt{2mE} \Rightarrow E < 0$

Let $k = i\sqrt{2m|E|} = i k_b$.

$$\Rightarrow \quad \frac{4k_b}{U_0} = \left(1 \pm e^{-k_b L}\right). \quad \begin{matrix} + = \text{even} \\ - = \text{odd} \end{matrix} \Big\} \text{ Parity}$$



$\Rightarrow$ there is always a bound state for the even parity solution.



$\Rightarrow$ a solution will exist if both have similar slopes at $k_b \to 0$. Let's expand $1 - e^{-k_b L} \sim k_b L$

$$\frac{4k_b}{U_0} = k_b L \Rightarrow (U_0)^{crit} = 4/L$$

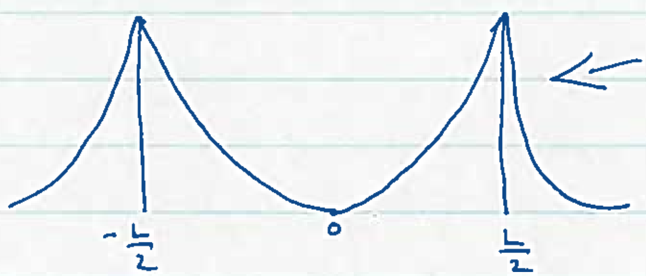
$\Rightarrow$ $U_0 > 4/L$ for a bound state to form of the odd parity solution. This is orthogonal to the condition for a transmission resonance to occur. It is for this reason that resonances and bound states should not be confused. Resonant states have $\Gamma = \infty$, whereas bound states have $\Gamma = 0$. For true bound states the reflection coefficient is not well-defined as $g(\pi)$ diverges when $\cot \delta_\ell = i$. Let us look at the states inside the well when $k = ik_b$.

When $k = i\kappa_b$ $\cos i\kappa_b L \Rightarrow \cosh \kappa_b L \Rightarrow$ the solutions inside the well are

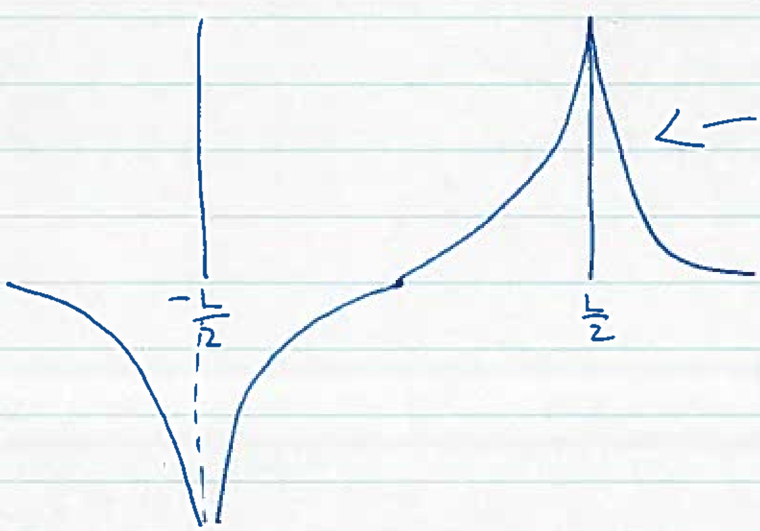
$$\cosh \kappa_b x = \frac{e^{-\kappa_b x} + e^{\kappa_b x}}{2}$$

$$\sinh \kappa_b x = \frac{e^{\kappa_b x} - e^{-\kappa_b x}}{2}.$$

This is what is expected. These are the $\pm$ combinations of the bound state solutions.



$\leftarrow$ + or bonding solution



$\leftarrow$ antibonding

This is all the physics of 2 δ-fcns.

Lecture 10: supplement

We showed last class that the transmission resonance condition is

$$\tan \kappa L = 4\kappa / U_0 .$$

For $-\frac{\pi}{2} \leq \kappa L \leq \frac{\pi}{2}$, the resonance condition holds as long as

$$\frac{4\kappa}{U_0} > \kappa L \Rightarrow \boxed{U_0^{crit} = \frac{4}{L}}$$

This condition is valid for small κ. However, $\tan \kappa L$ is a periodic function. $\Rightarrow$ $4\kappa/U_0$ will always find a branch of $\tan \kappa L$ to intersect. $\Rightarrow$ Strictly speaking there is always a solution to the resonance condition.

Now what do the $|T|^2 = 1$ states look like.

$$f(0) = T - 1 = i \sum_{\ell = 0,1} \frac{1}{\cot \delta_\ell - i}$$

At resonance $\cot \delta_0 = \cot \delta_1$. Let $\cot \delta_0 = \cot \delta_1 = \cot \delta$.

$$\Rightarrow T - 1 = i \frac{2}{\cot \delta - i}$$

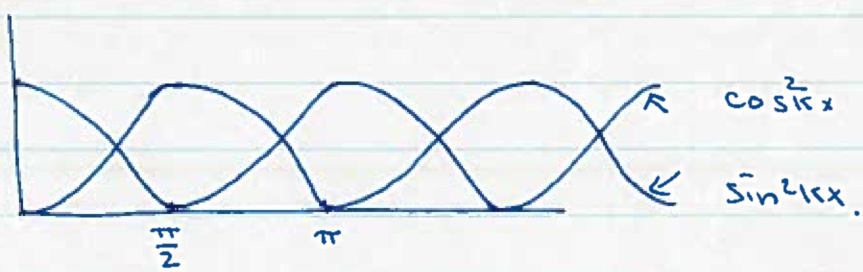
$$\Rightarrow T = \frac{\cot \delta + i}{\cot \delta - i} = \frac{1 + i \tan \delta}{1 - i \tan \delta} = e^{2i\delta}$$

The last equality follows from what we have shown before. $\Rightarrow$ $T \neq 1$ at resonance. But $|T|^2 = 1$.

$$\psi_L = e^{ikx} \quad \Big| \quad \begin{Bmatrix} \cos kx \\ \sin kx \end{Bmatrix} \quad \Big| \quad e^{2i\delta} e^{ikx}.$$

$$-\tfrac{L}{2} \qquad\qquad\qquad \tfrac{L}{2}$$

The states in between are linear combinations of e^{ikx} and e^{-ikx}. These states do correspond to the particle rattling around in $\pm k$ states and then leaving with a phase shift 2δ. The $-k$ part of the resonant state leads to reverse transmission which exactly cancels the Re^{-ikx} term of ψ_L. $\Rightarrow$ only e^{ikx} survives. That is, the two interfere destructively. $\Rightarrow$ they are 180° out of phase. The charge density between the two δ-fcn.'s looks like the following



The value of δ is determined by solving $4k/u_0 = \tan kL$, for k and then substituting back into either of

$$\cot\delta = \frac{4k/u_0 \pm \sin kL}{1 \pm \cos kL}.$$