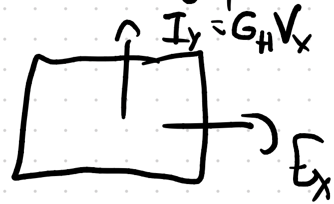
Lecture 23

Recap: Chern insulator

$$C = \frac{1}{2\pi}\oiint d^2k \; \underbrace{\mathrm{tr}\,\Omega_{xy}}_{\text{Berry curvature}} = \frac{1}{2\pi}\oint dk_x \frac{\partial}{\partial k_x} \mathrm{Im}\log\det W_{\frac{2\pi}{b}\leftarrow 0}(k_x)$$

We showed:

1. $C \in \mathbb{Z}$ – C can't change under perturbations if a gap remains open – topological invariant

2. 

$G_H = C\,\frac{e^2}{h}$ Hall conductance

3. $C \neq 0 \Rightarrow$ occupied states have no exponentially localized Wannier Functions

4) If we have inversion symmetry

$$(-1)^C = \prod_{\substack{k_* \in TRIM \\ n \text{ occupied}}} \xi_n(k_*)$$

$\xi_n(k_*) = \pm 1$ are the inversion eigenvalues

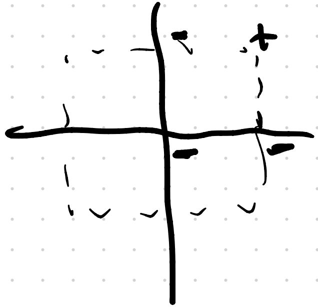
Chern insulator tight binding model

$$h(k) = \left(\Delta[1+\cos k_y b] + t_1 \cos k_x a\right)\sigma_z + t_2 \sin k_x a \, \sigma_y - t_2 \sin k_y b \, \sigma_x$$

$$B(I) = \sigma_z$$

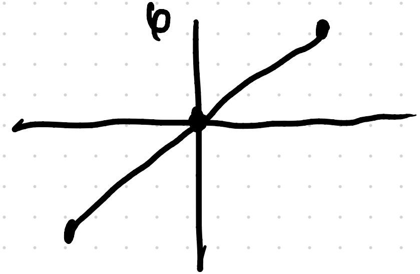
$$h(\Gamma) = (2\Delta + t_1)\sigma_z$$
$$h(X) = (2\Delta - t_1)\sigma_z$$
$$h(Y) = t_1 \sigma_z$$
$$h(M) = -t_1 \sigma_z$$

1) $2\Delta > t_1 > 0$

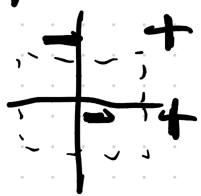


$(-1)^C = -1$

$|C| = 1$

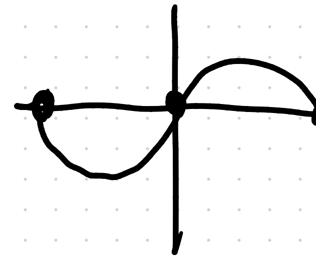


② $t_1 > 2\Delta > 0$



$(-1)^C = +1$

$C = 0$



$C = \pm 1$ or 0

Now - Time-reversal invariant systems

$\text{TRS} \rightarrow C = 0$

Time reversal symmetry : T antiunitary

$$P(-k) = T P(k) T^{-1}$$

$$\langle v|w\rangle = \langle Tw|Tv\rangle$$

$$W_{\frac{2\pi}{b}\leftarrow 0}(k_x) = \langle u_{n\,k_x,\frac{2\pi}{b}} | \prod_{k_y}^{\frac{2\pi}{b}\leftarrow 0} P(k_x,k_y) | u_{m\,k_x,0} \rangle$$

$$= \langle T u_{m\,k_x,0} | T \Big(\prod_{k_y}^{0\leftarrow -\frac{2\pi}{b}} P(k_x,k_y) \Big)^{\dagger} u_{n\,k_x,\frac{2\pi}{b}} \rangle$$

$$(i\sigma_y)^{\dagger}(i\sigma_y) = \sigma_0$$

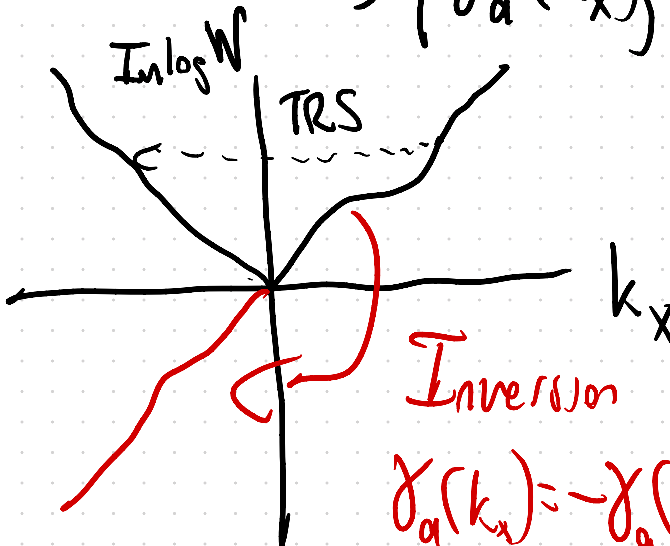
$$= B^{\dagger}_{m\ell,k_x 0}(T) \langle u_{m\,-k_x,0} | \prod_{k_y}^{0\leftarrow \frac{2\pi}{b}} P(-k_x,-k_y)\, u_{n\,-k_x,-\frac{2\pi}{b}} \rangle B_{k_x 0}(T)$$

$$W_{\frac{2\pi}{b}\leftarrow 0}(k_x) = B^{\dagger}(T)\, W^{T}_{\frac{2\pi}{b}\leftarrow 0}(-k_x)\, B(T)$$

eigenvalues

$W_{\frac{2\pi}{b}\leftarrow 0}(-k_x)$ has eigenvalues $\{e^{i\gamma_a(k_x)}\}$

$$\{e^{i\gamma_a(k_x)}\} \Rightarrow \{\gamma_a(k_x)\} = \{\underline{\gamma_a(-k_x)}\}$$



$$\gamma_a(k_x) = -\gamma_a(-k_x)$$

TRS - Wilson loop spectrum is symmetric under $k_x \to -k_x$

TRS: $\det W(k_x) = \det W(-k_x)$

$$C = \frac{1}{2\pi}\int_{-\pi/a}^{\pi/a} dk_x \frac{\partial}{\partial k_x} \operatorname{Im} \log \det W(k_x)$$

$$= \frac{1}{2\pi}\left[\int_{-\pi/a}^{0} dk_x \frac{\partial}{\partial k_x} \operatorname{Im}\log\det W(k_x) + \int_{0}^{\pi/a} dk_x \operatorname{Im}\log\det W(k_x)\right]$$

$$k_x \to -k_x$$

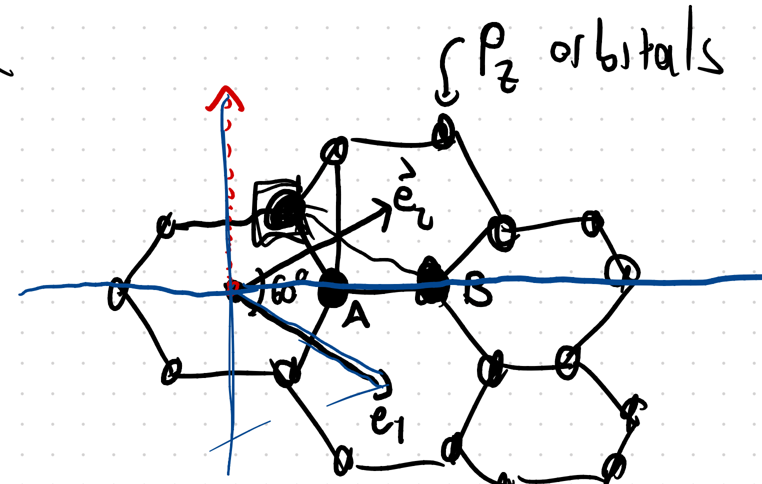
$$= \frac{1}{2\pi}\int_{0}^{\pi/a} dk_x \frac{\partial}{\partial k_x}\operatorname{Im}\log\det W(k_x) - \frac{\partial}{\partial k_x}\operatorname{Im}\log\det W(k_x)$$

$$= 0$$

Kane-Mele Model

$$\vec{e}_1 = \frac{a}{2}\left(\sqrt{3}\,\hat{x} - \hat{y}\right)$$

$$\vec{e}_2 = \frac{a}{2}\left(\sqrt{3}\,\hat{x} + \hat{y}\right)$$



$r_A = \frac{1}{3}\vec{e}_1 + \frac{1}{3}\vec{e}_2$

$r_B = \frac{2}{3}\vec{e}_1 + \frac{2}{3}\vec{e}_2$

C_{6z}: $\vec{e}_1 \to \vec{e}_2$

$e_2 \to \vec{e}_2 - \vec{e}_1$

$r_A \to \frac{1}{3}(e_2 + e_2 - e_1)$

$= \frac{2}{3}e_2 - \frac{1}{3}e_1$

$= r_B - \vec{e}_1$

$r_B \to r_A - e_1 + e_2$

+ TRS

+ Inversion

$I: (x, y, z=0) \to (-x, -y, z=0)$

$M_{1\bar{1}} = M_y$: $e_1 \leftrightarrow e_2$

$r_A \to r_A$

$r_B \to r_B$

$p\,6/mmm1'$ ← Timereversal symmetry

primitive hexagonal → p

C_{6z} → 6

Inversion → $/$

$M_{1\bar{1}}, M_{10}, M_{01}$ → mm

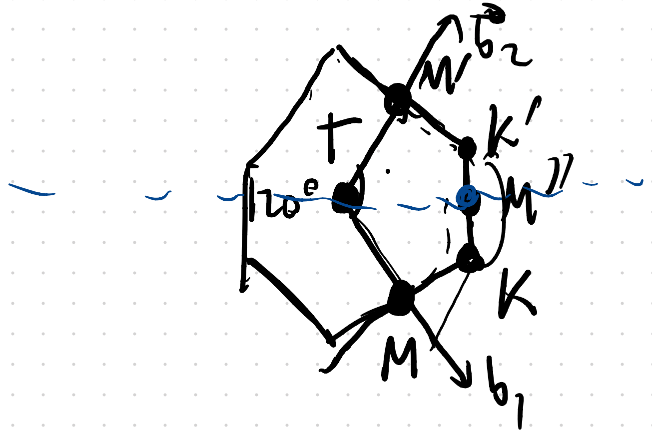
$$C_{6z}^2 = C_{3z}$$

$$C_{6z}^3 = C_{2z}$$

Reciprocal lattice

$$\vec{b}_1 = \frac{2\pi}{a}\left(\frac{1}{\sqrt{3}}\hat{x} - \hat{y}\right)$$

$$\vec{b}_2 = \frac{2\pi}{a}\left(\frac{1}{\sqrt{3}}\hat{x} + \hat{y}\right)$$



$$\Gamma = \vec{0}$$
$$M = \tfrac{1}{2}\vec{b}_1$$
$$M' = \tfrac{1}{2}\vec{b}_2$$
$$M'' = \tfrac{1}{2}(\vec{b}_1 + \vec{b}_2)$$

$\left.\right\}$ TRIM

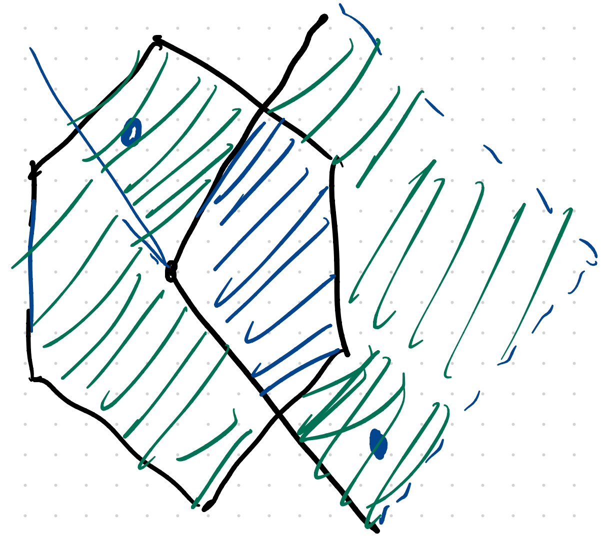
$$K = \tfrac{2}{3}\vec{b}_1 + \tfrac{1}{3}\vec{b}_2$$
$$K' = \tfrac{2}{3}\vec{b}_2 + \tfrac{1}{3}\vec{b}_1$$

$G_\Gamma = p6/mmm1'$

$G_{M''} \ni M_y, C_{2z}, I, TRS$

" $pmmm1'$

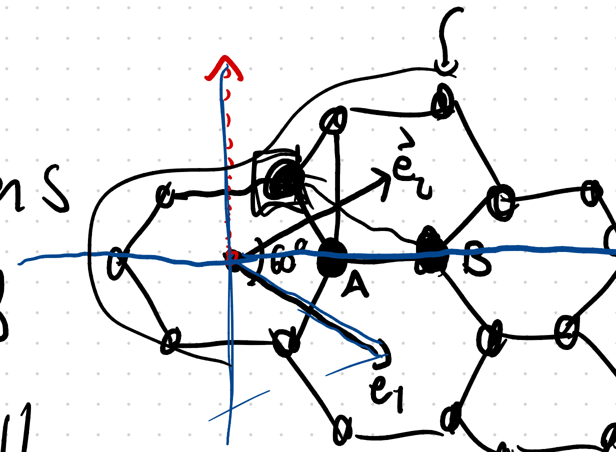
$G_K \ni C_{3z},\ C_{6z}M_y = p3m$



Tight-binding basis functions

$|\tau \sigma \vec{R}\rangle$ $\tau = A, B$

$\vec{R}$ unit cell



$\sigma = \uparrow / \downarrow$ spin

$R + r_A \to -R - r_A$

$= -R + r_B - e_1 - e_2$

Band representation:

$u_I | A \sigma \vec{R} \rangle = | B \sigma \, -R - e_1 - e_2 \rangle (-1)$ ← from $(-1)^\ell$ for p orbitals

$u_I | B \sigma \vec{R} \rangle = | A \sigma, R - e_1 - e_2 \rangle (-1)$

$$\boxed{B(I) = (-1)\, \tau_x \otimes \sigma_0}$$ Band rep matrix

$C_6 | A \sigma \vec{R} \rangle = | B \sigma' \, C_6 R - e_1 \rangle \left(e^{-\pi i/6 \, \sigma_z} \right)_{\sigma' \sigma} e^{-2\pi i/6}$ ← orbital rotation

$$C_6|B\sigma\vec{R}\rangle = |A\sigma' C_6R - e_1 + e_2\rangle \times \left(e^{-i\pi/6\,\sigma_z}\right)_{\sigma\sigma'} e^{-2\pi i/6}$$

spin rotation

$$\boxed{\begin{aligned} B(C_6) &= e^{-2\pi i/6}\,\tau_x \otimes e^{-i\pi/6\,\sigma_z} \\ B(C_3) = B(C_6)^2 &= e^{-2\pi i/3}\,\tau_0 \otimes e^{-i\frac{\pi}{3}\sigma_z} \\ B(C_2) = B(C_6)^3 &= e^{-i\pi}\,\tau_x \otimes e^{-i\frac{\pi}{2}\sigma_z} = i\tau_x \otimes \sigma_z \end{aligned}}$$

$$T|A\sigma\vec{R}\rangle = |A\sigma'\vec{R}\rangle(-i\sigma_y)_{\sigma'\sigma}$$

$$\boxed{B(T) = -i\,\tau_0 \otimes \sigma_y\, \mathcal{K}}$$