Lecture 24

HW 4 Due Today

HW5 – Due 12/11 ← update

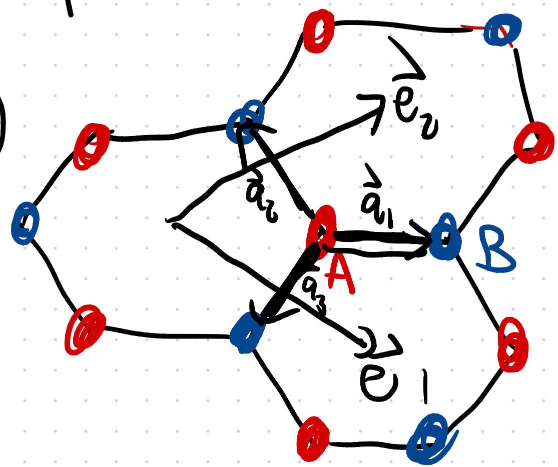
will be posted today

$p\,6/mmm\,1'$

$\vec{a}_1 = \vec{r}_B - \vec{r}_A = \frac{1}{3}(e_1 + e_2)$

$\vec{a}_2 = \frac{1}{3}(\vec{e}_2 - 2\vec{e}_1)$

$\vec{a}_3 = \frac{1}{3}(\vec{e}_1 - 2\vec{e}_2)$



$\vec{r}_A = \frac{1}{3}(\vec{e}_1 + \vec{e}_2)$

$\vec{r}_B = \frac{2}{3}(\vec{e}_1 + \vec{e}_2)$

$|\tau\,\sigma\,\vec{R}\rangle$ – tight-binding

basis is p-orbitals

$\tau = A, B$

$$B(I) = -\tau_x \otimes \sigma_0$$

$$B(T) = -i\, \tau_0 \otimes \sigma_y \mathcal{K}$$

$\sigma = \uparrow, \downarrow$

Nearest Neighbor tight-binding model

spin-independent

$$t = \langle B \sigma \vec{R} | H | A \sigma \vec{R} \rangle$$

$\{C_3 | \vec{e}_1\}$ symmetry: $t = \langle B\sigma\, -\vec{e}_1 | H | A \sigma \vec{0} \rangle$

$$= \langle B\sigma\, -\vec{e}_2 | H | A\sigma \vec{0}$$

TRS: $t = t^*$

Inversion: $t = \langle A\sigma\vec{R}|H|B\sigma\vec{R}\rangle$

$$h^{\tau\tau'}_{\sigma\sigma'}(\vec{k}) = \sum_R e^{-ik\cdot(R+r_\tau - r_{\tau'})}\delta_{\sigma\sigma'}\langle\tau\sigma R|H|\tau'\sigma'\vec{0}\rangle$$

$$= \begin{array}{cc} & \tau' \longrightarrow \\ \tau \downarrow & \begin{pmatrix} 0 & q(\vec{k}) \\ q^*(k) & 0 \end{pmatrix} \end{array} \otimes \sigma_0 \qquad q(k) = t\sum_{i=1}^{3} e^{-ik\cdot\vec{a}_i}$$

$$\vec{k} = k_1\vec{b}_1 + k_2\vec{b}_2$$

$$q(k) = t\left(e^{-\frac{2\pi i}{3}(k_1+k_2)} + e^{-\frac{2\pi i}{3}(k_2-2k_1)} + e^{-\frac{2\pi i}{3}(k_1-2k_2)}\right)$$

$$\mathcal{E}_\pm(k) = \pm|q(k)|$$

$\Gamma\ (0,0)$

$q(\Gamma) = 3t$

$h(\Gamma) = 3t\,\tau_x \otimes \sigma_0$

$\mathcal{E}_\pm(\Gamma) = \pm 3t$

$M:\ (k_1 = \frac{1}{2}, k_2 = 0)$

$q(M) = t\left[\frac{1}{2} - \frac{i\sqrt{3}}{2}\right]$

$h(M) = t\begin{pmatrix} 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} \\ \frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \end{pmatrix} \otimes \sigma_0$

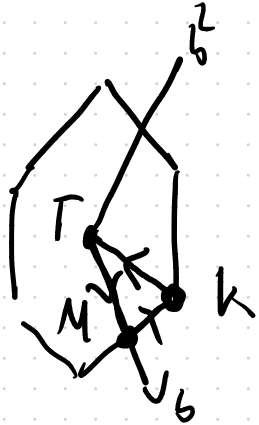
$\mathcal{E}_\pm(M) = \pm t$

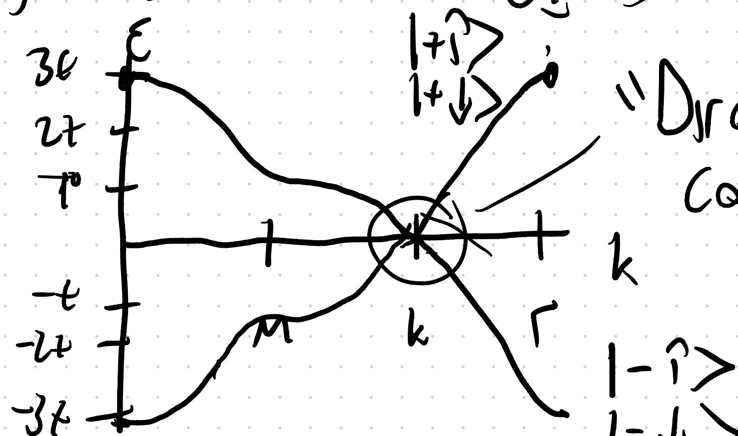
$K\ (k_1 = \frac{2}{3}, k_2 = \frac{1}{3})$

$q(K) = 0$

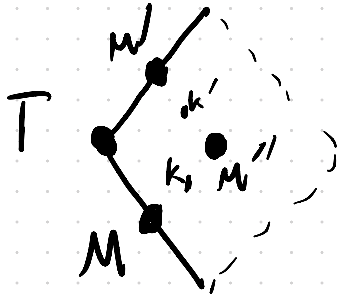
$h(K) = 0$

$\mathcal{E}_\pm(K) = 0$





"Dirac cone"



TRIMs T, M, M', M''

Inversion & TRS at the four TRIMs

$$V(\vec{b}) = \begin{pmatrix} e^{i\vec{b}\cdot\vec{r}_A} & \\ & e^{i\vec{b}\cdot\vec{r}_B} \end{pmatrix}$$

$$h(k+\vec{b}) = V^{\dagger}(\vec{b})\, h(k)\, V(\vec{b})$$

$$B(I) = -\tau_x \sigma_0$$

$$B^\dagger(I)\,h(k)\,B(I) = (\tau_x \otimes \sigma_0)\left[\begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix} \otimes \sigma_0\right] \tau_x \otimes \sigma_0$$

$$= \begin{pmatrix} 0 & q^*(k) \\ q(k) & 0 \end{pmatrix} \otimes \sigma_0 = h(-k)$$

TRS

$$(i\tau_0 \sigma_y \mathcal{K})\,h(k)\,(-i\tau_0\sigma_y\mathcal{K})^{-1} = h^*(k) = h(-k)$$

T: $h(\Gamma) = h(0) = 3t\,\tau_x \otimes \sigma_0$

$[B(I), h(0)] = 0$

$1 - \alpha k = 0 >$ - has inversion eigenvalues $+1$ for each spin

$$(+I, +I)$$

M: $\vec{k} = \frac{1}{2}\vec{b}_1$

$$B(I)\,h\left(\tfrac{1}{2}\vec{b}_1\right)B^\dagger(I) = h\left(-\tfrac{1}{2}b_1\right)$$

$$= h\left(\tfrac{1}{2}\vec{b}_1 - \vec{b}_1\right)$$

$$= V(b_1)\,h\left(\tfrac{1}{2}\vec{b}_1\right)V^\dagger(\vec{b}_1)$$

$$\left[V^\dagger(b_1)B(I),\, h\left(\tfrac{1}{2}\vec{b}_1\right)\right] = 0$$

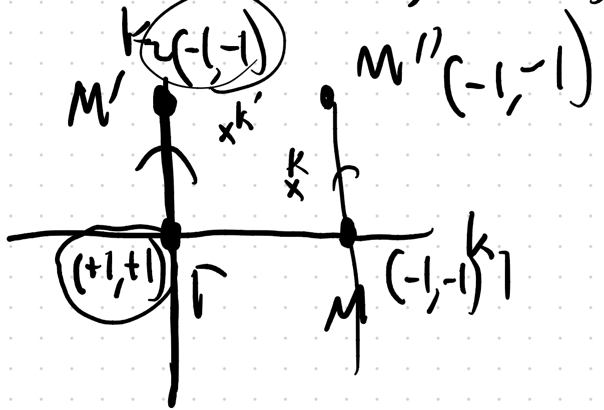
$$V^{\dagger}(b_1)\,B(I) = -\left[\begin{pmatrix} e^{-i b_1 \cdot r_A} & 0 \\ 0 & e^{-i b_1 \cdot r_B} \end{pmatrix} \otimes \sigma_0\right]\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \sigma_0\right]$$

$$= -\begin{pmatrix} 0 & e^{-i b_1 \cdot r_A} \\ e^{-i b_1 \cdot r_B} & 0 \end{pmatrix} \otimes \sigma_0$$

$$= \begin{pmatrix} 0 & -e^{-2\pi i/3} \\ -e^{2\pi i/3} & 0 \end{pmatrix} \otimes \sigma_0 = \frac{h(\frac{1}{2} b_1)}{t}$$

Negative energy bands have

(-1, -1) inversion eigenvalues



$$h(k)\vec{u}^{\sigma}_{-} = \varepsilon_{-}\vec{u}^{\sigma}_{-}$$

$$W^{\sigma\sigma'}_{1\leftarrow 0}(k_1) = \vec{u}^{\sigma}_{-k_1 0} \cdot V(\vec{b}_0) \prod_{k_2}^{1\leftarrow 0} P(k_1, k_2) \cdot u^{\sigma'}_{-k_1 0}$$

When $k_1 = k_1^* = 0, \pi$, we know

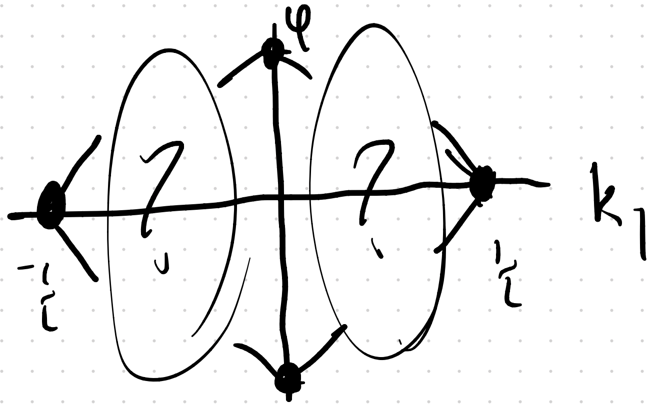
$$W^{\sigma\sigma'}_{1\leftarrow 0}(k_1^*) = B^-_{(k_1^*,0)}(I)\, W^\dagger_{i\leftarrow 0}(k_1^*)\, B^-_{(k_1^*,\pi)}(I)\, W_{i\leftarrow 0}(k_1^*)$$

proportional to identity σ_0

$(e^{i\pi}, e^{-i\pi})$

$$= B^-_{(k_1^*,0)}(I)\, B^-_{(k_1^*,\pi)}(I) = \begin{cases} -\sigma_0, & k_1 = 0 \\ +\sigma_0 & k_1 = \frac{1}{2} \end{cases}$$

(e^{i0}, e^{i0})

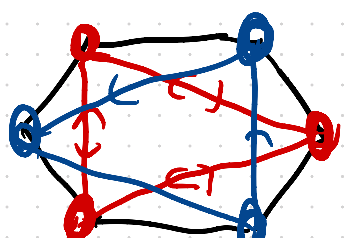


$e^{\pm i k \cdot e_1}$ $\quad e^{\pm i k \cdot e_2}$

$e^{\pm i k \cdot (e_2 - e_1)}$

Kane-Mele: Spin-orbit coupling

$$\delta h(k) = \lambda \left[\sin 2\pi k_1 - \sin 2\pi k_2 + \sin 2\pi (k_2 - k_1) \right] \tau_z \otimes \sigma_z$$



Next-nearest neighbor spin-dependent hopping

$$B(I)\,\delta h(k)\,B^{\dagger}(I) = -\delta h(k) = \delta h(-k) \checkmark\checkmark$$

TRS $$(-i\sigma_y)\,\delta h^{*}(k)\,(i\sigma_y) = -\delta h(k) = \delta h(-k) \checkmark\checkmark$$

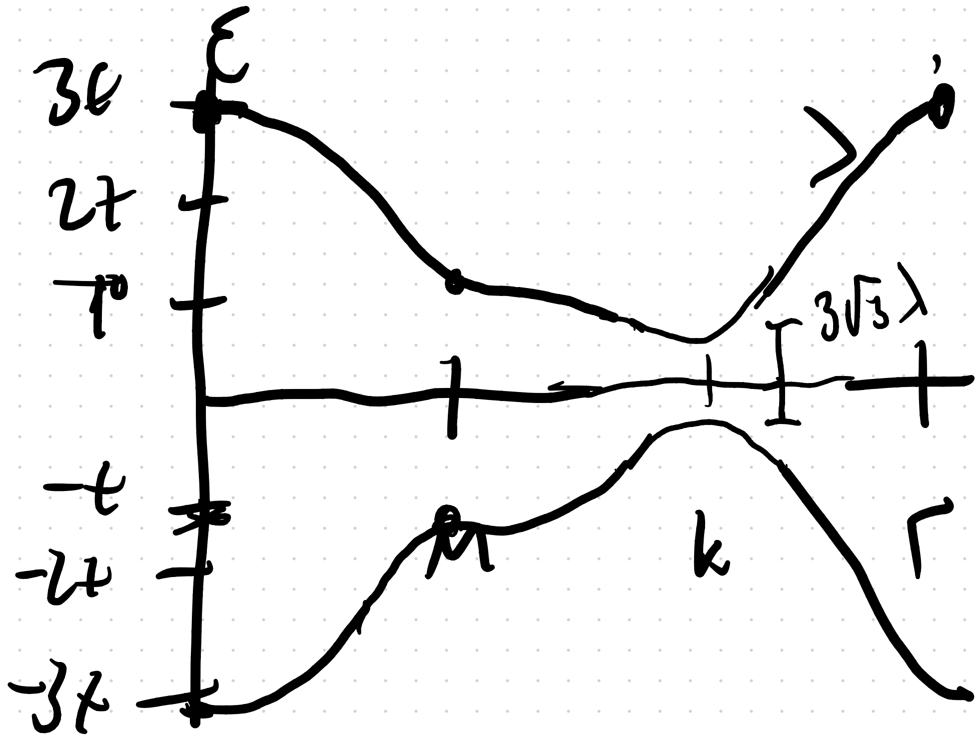
Effects on the spectrum:

$$\delta h(\Gamma) = 0$$

$$\delta h(M) = 0$$

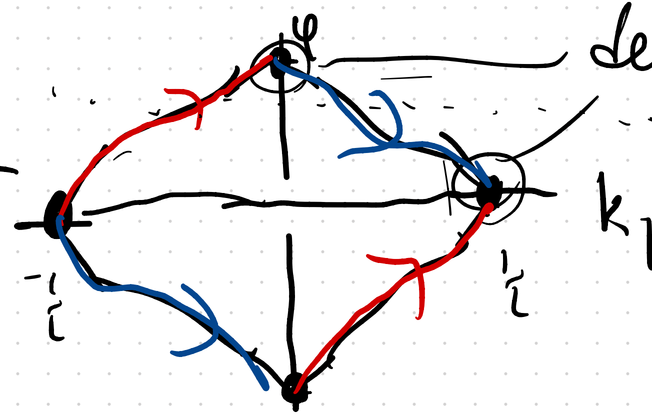
$$\delta h(K) = \delta h\left(k_1 = \tfrac{2}{3}, k_2 = \tfrac{1}{3}\right) = \frac{-3\sqrt{3}}{2}\lambda\, \tau_z \otimes \sigma_z$$ ← has espectrales $\pm \frac{3\sqrt{3}}{2}\lambda$

Spectrum of $h(k) + \delta h(k)$ ← Kane-Mele



Gapped → Wilson loop:

$\nu = 1$



degeneracy from TRS

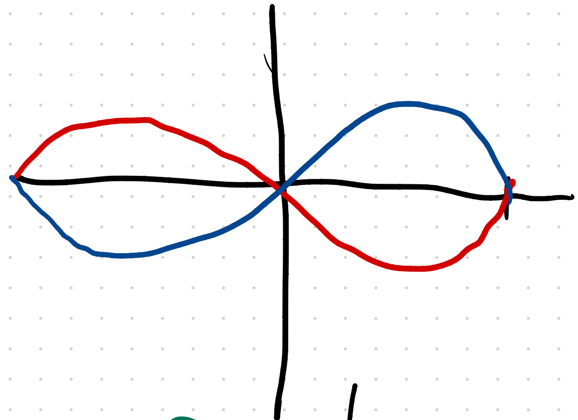
$C = 0$ - No chern number

"helical winding"

Quantum Spin-Hall insulator

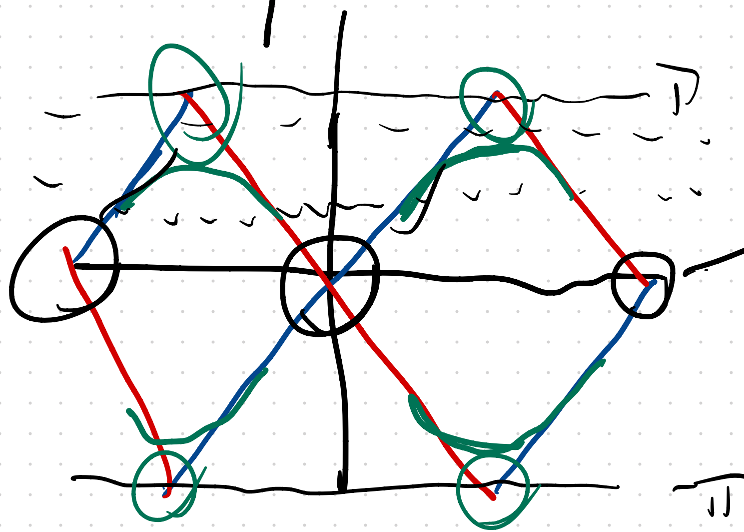
ν - helical winding number

$\nu = 0$



"Trivial"

$\nu = 2$



Protected by TRS

$\downarrow$

$\nu = 0$

Without closing an energy gap

ν is only invariant modulo 2

ν mod 2 - the $\mathbb{Z}_2$ invariant

Kane-Mele invariant

- Time-reversal symmetry needed

$$(-1)^{\nu} = (-1)^{\sum_{\text{trim pts}} \frac{1}{2} n}$$