Lecture 3

Announcements: - HW I posted, due 9/19 ← Thurs

- Office Hours start next week, Weds 9/11 at 4pm via zoom link on course website

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Recap: Groups, subgroups, cosets → First Isomorphism Theorem

$\phi: G \to K$ a homomorphism $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$ for $g_1, g_2 \in G$

Then: $\operatorname{Im}\varphi \leq K$

$\operatorname{Ker}\varphi \trianglelefteq G$

$G/\ker\varphi \cong \operatorname{Im}\varphi$ } there exists an invertible homomorphism $\psi: G/\ker\varphi \leftrightarrow \operatorname{Im}\varphi$

One last general point about quotient groups

let $H \triangleleft G$ be a normal subgroup

$$G = \bigcup_{i=0}^{n-1} Hg_n \qquad g_0 \equiv E_G$$

$G/H = \{H, Hg_1, Hg_2, \ldots Hg_{n-1}\}$ is a group (quotient group)

In <u>some</u> cases we can relate elements of G/H to elements in G: there exists a homomorphism

$$i : G/H \to G$$

$$i(Hg_i) = \tilde{g}_i \in Hg_i$$

$$i(H) = E$$

If i exists then $\operatorname{Im} i = \{E, \tilde{g}_1, \tilde{g}_2, \cdots \tilde{g}_{n-1}\} \leq G$

and from the 1st isomorphism theorem, $\operatorname{Im} i \cong G/H$

furthermore: $\tilde{g}_i \in Hg_i \rightarrow Hg_i = H\tilde{g}_i$

so we can use $\{\tilde{g}_i\}$ as coset representatives

$$G = \bigcup_{i=0}^{n-1} H\tilde{g}_i = H(\operatorname{Im} i)$$ ← every element of

G can be written uniquely as $g = hk$

$h \in H \quad k \in \operatorname{Im} i = K$

when this is possible, we say G is a semidirect product $G = H \rtimes K$ $H \trianglelefteq G$

$K \leq G$

Example: The group of rigid transformations of 3D space

Euclidean group $\mathbb{E}(3)$: - translation

- rotation

- reflection

$\{R|\vec{v}\} = g \in \mathbb{E}(3)$

"Seitz symbol" for g

$R \in O(3)$ rotation or reflection

$\vec{v} \in \mathbb{R}^3$ a translation

Action on points in space

$$\{R|\vec{v}\}\vec{x} = [R]\cdot\vec{x} + \vec{v}$$

$[R]$ - 3x3 Matrix that implents $R \in O(3)$)

Multiplication in $E(3)$

$$g_1 = \{R_1|\vec{v}_1\}$$

$$g_2 = \{R_2|\vec{v}_2\}$$

$$\begin{aligned}(g_1 g_2)\vec{x} &= g_1(g_2\vec{x}) \\ &= \{R_1|\vec{v}_1\}\left([R_2]\vec{x} + \vec{v}_2\right) \\ &= [R_1]\left([R_2]\vec{x} + \vec{v}_2\right) + \vec{v}_1 \\ &= [R_1 R_2]\vec{x} + \left(\vec{v}_1 + [R_1]\vec{v}_2\right)\end{aligned}$$

$$= \{R_1R_2 \mid v_1 + R_1\vec{v}_2\}\vec{x}$$

$$\boxed{\{R_1 | v_1\}\{R_2 | v_2\} = \{R_1R_2 \mid \vec{v}_1 + R_1\vec{v}_2\}}$$

- inverses: $g = \{R|\vec{v}\}$

$$g^{-1} = \{R|v\}^{-1} = \{R^{-1} \mid -R^{-1}v\}$$

check: $gg^{-1} = \{R|v\}\{R^{-1}|-R^{-1}v\}$

$$= \{E|\vec{0}\}$$ - identity in $E(3)$

① $O(3) \leq E(3)$

$O(3) \cong \{ \underbrace{\{R|\vec{0}\}} \mid R \in O(3) \}$

(2) $\mathbb{R}^3 \triangleleft E(3)$

$-\mathbb{R}^3 < E(3)$ b/c $\{\{E|\vec{v}\} \mid \vec{v} \in \mathbb{R}^3\} \cong \mathbb{R}^3$

to see that its normal, we need to show

$g = \{R|\vec{d}\}$ is $g\{E|\vec{v}\}g^{-1} = \{E|\vec{v}'\}$

$\left(\{R|\vec{d}\}\{E|\vec{v}\}\right)\{R^{-1}|-R^{-1}\vec{d}\} = \{R|\vec{d}+R\vec{v}\}\{R^{-1}|-R^{-1}\vec{d}\}$

$= \{E|R\vec{v}\} \in \mathbb{R}^3$ //

Finally: $\{R|\vec{v}\} = \{E|\vec{v}\}\{R|\vec{0}\}$

$$\Rightarrow \mathbb{E}(3) = \mathbb{R}^3 \rtimes O(3)$$

to make contact w/ earlier notation?

$$H = \{\{E|\vec{v}\} \,|\, \vec{v} \in \mathbb{R}^3\}$$

$$G/H = \{H\{R|\vec{0}\} \,|\, R \in O(3)\}$$

$$i(H\{R|\vec{0}\}) = \{R|\vec{0}\}$$

Non example: $G = \left\langle \{E|a\hat{z}\}, \{C_{2z}|\tfrac{a}{2}\hat{z}\} \right\rangle < \mathbb{E}(3)$

$\langle \ldots \rangle$ - the group generated by ...

C_{2z} - 180° rotation about z-axis

$$\{C_{2z} | \tfrac{a}{2}\hat{z}\} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ -y \\ z+\frac{a}{2} \end{pmatrix}$$

$T \triangleleft G \qquad T = \{ \{E | na\hat{z}\} \mid n \in \mathbb{Z} \}$

$$\{C_{2z} | \tfrac{a}{2}\hat{z}\} \{E | na\hat{z}\} \{C_{2z} | -\tfrac{a}{2}\hat{z}\} = \{E | na\hat{z}\}$$

but note: $\{C_{2z} | \frac{a}{2}\hat{z}\}^2 = \{E | a\hat{z}\}$

$$G = T \cup T\{C_{2z} | \tfrac{a}{2}\hat{z}\}$$

$$i(T) = \{E|0\}$$

$$i(T\{C_{2z}|\tfrac{a}{2}\hat{z}\}) = \{C_{2z}|(n+\tfrac{1}{2})a\hat{z}\}$$

but $\{C_{2z}|(n+\tfrac{1}{2})a\hat{z}\}^2 = \{E|(2n+1)a\hat{z}\} \neq \{E|0\}$

So G is <u>not</u> a semidirect product

(if this were a semidirect product, I coud write $G = T \cup T\tilde{g}$ with $\tilde{g}^2 = E$ $\quad \{E, \tilde{g}\} \simeq G/T$)

How do we use groups in Condensed matter physics

$$H = \frac{p^2}{2m} + V(x) + \ldots$$ group G of symmetries $g \in G$

$$\vec{x} \to \vec{x}' = g\vec{x}$$

$$\vec{p} \to \vec{p}' = g\vec{p}$$

$$\psi'(x) \to \psi(g^{-1}x)$$

$$H' = H$$

$$G \ni g \to \rho(g) \in U(V)$$

V - QM Hilbert space

$|\psi\rangle \in V$

$U(V)$ - group of unitary operators

operator

on V

$$\rho^\dagger(g)\hat{X}\rho(g) = \hat{X}'$$

$$\rho^\dagger(g)\hat{p}\rho(g) = \hat{p}'$$

$$|\psi'\rangle = \rho(g)|\psi\rangle$$

ρ should be a group homomorphism

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$$

Define: a (unitary) representation of a group G is:

- a vector space V
- a homomorphism $\rho : G \to U(V)$

group of unitary operators/matrices on V

V is called the representation space

$\rho(g)$ is called the representative of g

Example: representation of $\mathbb{R}^3$ (translation group) on the hilbert space $V = \{\text{square integrable wavefunctions}\}$

$$\mathbb{R}^3 \ni \vec{v} \longrightarrow \rho(\vec{v}) = e^{-i\vec{p}\cdot\vec{v}/\hbar}$$

$$\rho(\vec{v}_1+\vec{v}_2) = e^{-i\vec{p}\cdot(v_1+v_2)/\hbar} = e^{-i\vec{p}\cdot v_1/\hbar} e^{-i\vec{p}\cdot v_2/\hbar} = \rho(\vec{v}_1)\rho(\vec{v}_2)$$

$$\rho^\dagger(\vec{v})\hat{x}\rho(\vec{v}) = e^{i\vec{p}\cdot v/\hbar}\hat{x}e^{-i\vec{p}\cdot v/\hbar} = \hat{x} + \frac{i}{\hbar}[v\vec{p}, \hat{x}]$$

$$= \hat{x} + \vec{v}$$

Less trivial example: $SO(2)$

$SU(2) \ni (\hat{n}, \theta)$ $\quad \theta \in [0, 4\pi)$

$\hat{n}$: a unit vector in 3d

2x2 identity; vector of Pauli matrices

$$(\hat{n},\theta) \xrightarrow[\rho_{1/2}]{} \cos\frac{\theta}{2}\,\sigma_0 + i\sin\frac{\theta}{2}\,\hat{n}\cdot\vec{\sigma}$$

$$\left.\begin{aligned}(\hat{n},\theta=0) &= E\\ (\hat{n},\theta=2\pi) &= \bar{E}\end{aligned}\right\}$$

$\bar{E}$ not identity

$\bar{E}^2 = E$

defining representation of $SU(2)$

$\ker \rho_{1/2} = \{E\}$

$\mathrm{Im}\,\rho_{1/2} \cong SU(2)$

$$\det\begin{pmatrix}-1 & \\ & -1\end{pmatrix} = +1$$

Other representations:

Spin $l=1$ representation ρ_1:

$$L_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \xrightarrow{\text{unitary transformation}} L_x' = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$$

$$L_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \longrightarrow L_y' = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \longrightarrow L_z' = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\rho_1((\hat{n},\theta)) = e^{-i\,\hat{n}\cdot\vec{L}\,\theta}$$

$$\mathrm{Im}\,\rho_1 \cong SO(3)$$

$$\mathrm{Ker}\,\rho_1 = \{E, \bar{E}\}$$

"SU(2) is the double cover of SO(3)"

1st Isomorphism theorem: $\boxed{SO(3)} \cong SU(2)/\{E,\bar{E}\}$

Two representations $\rho : G \to U(V)$
$\sigma : G \to U(V)$

are equivalent $\rho \cong \sigma$ if there is some unitary matrix A s.t.

$$A\rho(g)A^\dagger = \sigma(g) \text{ for all } g$$

$$\boxed{[e_i, e_j] = i\varepsilon_{ijk} e_k}$$

$$\sigma_i \to U\sigma_i U^\dagger$$