Lecture 5

Reminders: - HW I is due 9/19

- Office hrs start 9/11 @ 4pm via zoom

Recap: $\rho: G \to U(V)$ is an <u>irreducible representation</u> if it has no nontrivial invariant subspaces $\to$ we can't block-diagonalize every $\rho(g)$ simultaneously

Schur's Lemma: ① $\rho_1: G \to U(V_1)$
$\rho_2: G \to U(V_2)$ both irreducible

$H: V_1 \to V_2$ $\quad H\rho_1(g) = \rho_2(g) H$

then $H = 0$ or H is invertible

② $\rho_1 = \rho_2 = \rho$ $\quad [H, \rho(g)]$ forall $g \in G$

then $H = \lambda \, Id_V$

Part 2.5: $\rho_1 : G \to U(V_1)$
$\rho_2 : G \to U(V_2)$ irreducible

$H : V_1 \to V_2$ $\quad \rho_2(g) H = H \rho_1(g)$

"intertwiner"

then H is invertible $\Rightarrow$ $\rho_1 \cong \rho_2$ are unitarily equivalent

Pf: consider $H^\dagger = (H^*)^T$

$H^\dagger : V_2 \to V_1$

$$(\rho_2(g^{-1}) H)^\dagger = (H \rho_1(g^{-1}))^\dagger = \rho_1(g^{-1})^\dagger H^\dagger = \rho_1(g) H^\dagger$$

$$\parallel$$

$$H^\dagger \rho_2(g)$$

Schur's lemma: either $H^\dagger = 0$ or $H^\dagger$ is invertible

Consider $H^\dagger H : V_1 \to V_1$ $\qquad H^\dagger H \rho_1(g) = H^\dagger \rho_2(g) H$

$$= \rho_1(g) H^\dagger H$$

$$[\rho_1(g), H^\dagger H] = 0$$

$\downarrow$ Schur's lemma pt 2

$$H^\dagger H = \lambda \, Id_{V_1}$$

$$\boxed{H^\dagger = \lambda H^{-1}} \to \quad U \equiv \frac{1}{\sqrt{\lambda}} H$$

$$U^\dagger = \frac{1}{\sqrt{\lambda}} H^\dagger = \sqrt{\lambda} H^{-1} = U^{-1}$$ so U is unitary

$$\boxed{U^\dagger \rho_2(g) U = \sqrt{\lambda} H^{-1} \rho_2(g) \frac{H}{\sqrt{\lambda}} = H^{-1} \rho_2(g) H = \rho_1(g)}$$

contrapositive if $H\rho_1(g) = \rho_2(g)H$ and

if $\underline{\rho_1 \not\cong \rho_2}$ $\rightarrow$ $H = 0$

Character Theory:

- easy way to tell when representations are unitarily equivalent
- tell when a representation is irreducible
- count/enumerate all the irreps of a group

let $\rho: G \rightarrow U(V)$ a representation of G

Def <u>character</u> χ_ρ is a function

$$\chi_\rho: G \rightarrow \mathbb{C}$$
$$\chi_\rho(g) = tr\left[\rho(g)\right]$$

① If ρ_1 & ρ_2 are equivalent reps $\left(U\rho_1(g)U^\dagger = \rho_2(g)\right.$ for all $\left.g \in G\right)$

then $\chi_{\rho_2}(g) = \mathrm{tr}[\rho_2(g)]$

$$= \mathrm{tr}[U\rho_1(g)U^\dagger]$$

$$= \mathrm{tr}[\rho_1(g)] = \chi_{\rho_1}(g)$$

$$\rho_1 \cong \rho_2 \Rightarrow \chi_{\rho_1} = \chi_{\rho_2}$$

(reps w/ distinct characters are inequivalent)

② ρ a representation $\quad g_2 = g g_1 g^{-1}$ (g_1 & g_2 are conjugate)

$$\chi_\rho(g_2) = \mathrm{tr}[\rho(g_2)]$$

$$= \mathrm{tr}\left[\rho(g g_1 g^{-1})\right] = \mathrm{tr}\left[\rho(g)\rho(g_1)(\rho(g))^{-1}\right]$$

$$= \mathrm{tr}\left[\rho(g_1)\right] = \chi_\rho(g_1)$$

characters are invariant under conjugation:

all elements of the same conjugacy class

$$C_g = \{ g' \in G \mid g' = g_\alpha g g_\alpha^{-1} \text{ for some } g_\alpha \in G \}$$

have the same character as g

$\rightarrow$ Characters are constant on conjugacy classes (we call these "class functions")

(3) $\rho \cong \rho_1 \oplus \rho_2$ $\qquad \rho(s) = \left(\begin{array}{c|c} \rho_1(s) & 0 \\ \hline 0 & \rho_2(s) \end{array}\right)$

$$\chi_{\rho_1 \oplus \rho_2}(s) = \mathrm{tr}\left[\left(\begin{array}{c|c} \rho_1(s) & 0 \\ \hline 0 & \rho_2(s) \end{array}\right)\right]$$

$$= \mathrm{tr}[\rho_1(s)] + \mathrm{tr}[\rho_2(s)]$$

$$= \chi_{\rho_1}(s) + \chi_{\rho_2}(s)$$

$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$ $\leftarrow$ the character of a reducible rep is the sum of characters of its irreducible components

Example: the group D_2 from HW1

$$D_2 = \{E, C_{2x}, C_{2y}, C_{2z}\}$$

vector representation: $\rho_V(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\rho_V(C_{2x}) = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$$

Abelian group - each element is its own conjugacy class

$$\rho_V(C_{2y}) = \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$$\rho_V(C_{2z}) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$$

	E	C_{2x}	C_{2y}	C_{2z}
χ_{e_V}	3	-1	-1	-1

ρ_V: ~~irreducible~~ or reducible

three invariant subspaces

$$\left\{\begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ b_2 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ 0 \\ b_3 \end{pmatrix}\right\}$$

$$\rho_V = \rho_{B_1} \oplus \rho_{B_2} \oplus \rho_{B_3}$$

Aside: for a 1d representation

irreps

	E	C_{2x}	C_{2y}	C_{2z}
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1

conjugacy classes

Characters

$\rho(g) = 1 \times 1$ matrix - #

$\mathrm{tr}\, \rho(g) = \rho(g)$

$\chi_\rho(g)$

Character tables

Aside: $\rho(E) = \mathrm{Id}$

$\chi_\rho(E) = \mathrm{tr}(\mathrm{Id}) = \dim \rho$

Schur's Lemma → Wonderful Orthogonality Relations / Schur Orthogonality relations

Let G be a finite group, $\rho_1 : G \to U(V_1)$
$\rho_2 : G \to U(V_2)$

both irreducible

and A <u>any</u> arbitrary $\dim V_2 \times \dim V_1$ matrix
$A : V_1 \to V_2$

trick: summing over all group elements

$$A_G = \sum_{g\in G} \rho_2(g^{-1}) A \rho_1(g)$$

then $$\underline{A_G \rho_1(g')} = \sum_{g\in G} \rho_2(g^{-1}) A \rho_1(g) \rho_1(g')$$

$$= \sum_{g\in G} \rho_2(g^{-1}) A \rho_1(gg')$$

$$g'' = gg'$$

$$= \sum_{g''\in G} \rho_2(g' g''^{-1}) A \rho_1(g'')$$

$$g^{-1} = (g'' g'^{-1})^{-1} = g' g''^{-1}$$

$$= \underline{\rho_2(g') A_G}$$

So A_G satisfies the assumptions of Schur's lemma part (1)

so then A_G is $\begin{cases} A_G = 0 & (\text{if } \rho_1 \not\cong \rho_2) \\ A_G = \lambda \, \mathrm{Id} & \rho_1 = \rho_2 \end{cases}$

Apply this to a special choice

$$A = E_{ij} = \begin{array}{cl} \text{jth column} & \\ \begin{pmatrix} 0 & 0 & \cdots & - \\ 0 & & 1 & \\ \vdots & & & \\ \vdots & & & \end{pmatrix} & \text{ith row} \end{array}$$

$$[E_{ij}]^{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$$

$$[E_{ij}]^{\mu\nu}_G = \sum_{\alpha\beta}\sum_{g\in G} [\rho_2(g^{-1})]^{\mu\alpha}[E_{ij}]^{\alpha\beta}[\rho_1(g)]^{\beta\nu}$$

$$= \sum_{g\in G} [\rho_2(g^{-1})]^{\mu i}[\rho_1(g)]^{j\nu} = \begin{cases} 0 \text{ if } \rho_1 \neq \rho_2 \\ \lambda\delta_{\mu\nu} \text{ if } \rho_1 = \rho_2 \end{cases}$$

to find λ let $\rho_1 = \rho_2$ and take the trace of both sides

LHS $\sum_{\mu} \sum_{g \in G} [\rho_1(g^{-1})]^{\mu i} [\rho_1(g)]^{j\mu}$

$$= \sum_{g \in G} \delta_{ij} = |G| \delta_{ij}$$

RHS $\sum_{\mu} \lambda \delta_{\mu\mu} = \lambda \dim \rho_1$ $\quad \lambda = \frac{|G|}{\dim \rho_1} \delta_{ij}$

Schur Orthogonality Formula

$$\sum_{g \in G} [\rho_2(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\dim \rho_1} \delta_{ij} \delta_{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

$$\left[\quad (\rho_1 = U \rho_2 U^\dagger \right.$$

$$\sum_{g\in G} [\rho_2(g^{-1})]^{\mu i} [U\rho_2(g)U^\dagger]^{j\nu}$$

$$= \sum_{g\in G} [\rho_2(g^{-1})]^{\mu i} U^{j\alpha} \rho_2(g)^{\alpha\beta} U^\dagger_{\beta\nu}$$

$$= U^{j\alpha} U^\dagger_{\beta\nu} \left[\frac{|G|}{\dim \rho_1} \delta_{i\alpha} \delta_{\mu\beta} \right]$$

$$\left. = \frac{|G|}{\dim \rho_1} U^{ji} U^\dagger_{\mu\nu} \quad \right]$$

$$\sum_{\mu\nu}\sum_{g\in G}\left[\rho_2(g^{-1})\right]^{\mu\mu}\left[\rho_1(g)\right]^{\nu\nu} = \begin{cases} 0 & \text{if } \rho_1\not\cong\rho_2 \\ \dfrac{|G|}{\cancel{d_{\text{irrep}}}}\cancel{\sum\delta_{\mu\nu}\delta_{\mu\nu}} \quad 1 & \text{if } \rho_1\cong\rho_2 \end{cases}$$

$$\parallel$$

$$\sum_{g\in G}\chi_{\rho_2}(g^{-1})\chi_{\rho_1}(g)$$

$$\parallel$$

$$\sum_{g\in G}\chi^*_{\rho_2}(g)\chi_{\rho_1}(g)$$

$$\Rightarrow \quad \frac{1}{|G|}\sum_{g\in G}\chi^*_{\rho_2}(g)\chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1\not\cong\rho_2 \\ 1 & \text{if } \rho_1\cong\rho_2 \end{cases}$$

$$|||$$

$$\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle$$

Under this inner product, irreducible characters are orthonormal

$\Rightarrow$ we can use characters to figure out how to decompose a representation ρ into irreps

$$\rho \cong \underbrace{\rho_1 \oplus \rho_1 \oplus \cdots}_{n_1} \oplus \underbrace{\rho_2 \oplus \rho_2 \oplus \cdots}_{n_2} \oplus \rho_3 \oplus \cdots$$

$$\cong \bigoplus_{i=1}^{N} n_i \rho_i \qquad n_i - \text{multiplicity of } \rho_i$$

in the decomposition of ρ

$$\chi_\rho = \sum_{i=1}^{N} n_i \chi_{\rho_i}$$

$$\left\langle \chi_{\rho_j}, \chi_\rho \right\rangle = \sum_{i=1}^{N} n_i \left\langle \chi_{\rho_j}, \chi_{\rho_i} \right\rangle = n_j$$

Ex: D_2

irreps

	E	C_{2x}	C_{2y}	C_{2z}
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1
A	1	1	1	1

← conjugacy classes

Characters

Character tables

Characters are constant on conjugacy classes

$\to$ # of irreps $\leq$ # of conjugacy

Clases