Lecture 2

Logistics: Office Hrs Mondays 4pm-5pm

Start 1/22

Recap: We introduced groups G - sets w/ associative "multiplication", identity element, inverses

Ex: Bravais lattice $T = \{ n\vec{t}_1 + m\vec{t}_2 + \vec{t}_3 \mid \vec{t}_1, \vec{t}_2, \vec{t}_3$ linearly independent, $n, m, l \in \mathbb{Z} \}$

- closed under "+"
- identity: $\vec{0} = 0\vec{t}_1 + 0\vec{t}_2 + 0\vec{t}_3$

- inverses $(n\vec{t}_1 + m\vec{t}_2 + m\vec{t}_3)^{-1} = (-n)\vec{t}_1 + (-m)\vec{t}_2 + (-\cancel{l})\vec{t}_3$

is a group

$H \subset G$ is a subgroup if H is a group and H is a subset of G

$$Hg_1 = \{hg_1 \mid h \in H\}$$

$$G = H \cup Hg_1 \cup Hg_2 \cdots \cup Hg_{n-1}$$

$n = |G : H|$ is called the <u>index</u> of H in G

Define conjugation by $g_1 \in G$

$$g \to \underline{g_1 g g_1^{-1}}$$

we say that two elements $g_2, g_3 \in G$ are <u>conjugate</u> if there exists $g \in G$ s.t. $g_2 = g g_3 g^{-1}$

C_{g_2} - conjugacy class of g_2

$= \{$all elements conjugate to $g_2\}$

Given a subgroup $H \subset G$ we can conjugate H by elements $g \in G$

$H \to gHg^{-1}$

Define: H is called a normal subgroup if $H = gHg^{-1}$ for all $g \in G$

$Hg = gH$

right cosets

left cosets

$H \triangleleft G$

Coset decomposition $G = H \cup Hg_1 \cup Hg_2 \cup \dots \cup Hg_{n-1}$

if $H \triangleleft G$ then the set of cosets $\{H, Hg_1, \dots, Hg_{n-1}\}$ forms a group!

$$Hg_i Hg_j = \{hg_i w g_j \mid h, w \in H\}$$

$$H \trianglelefteq G \Rightarrow g_i H = Hg_i$$

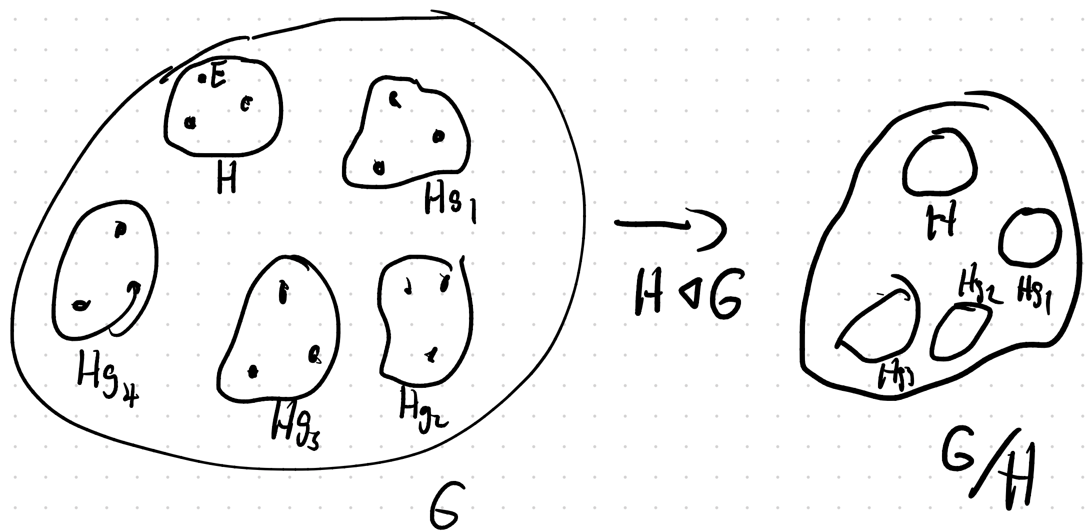
$$H(g_i H)g_j = HHg_i g_j = Hg_i g_j$$ - For normal subgroups, the product of two cosets is also a coset

$$HHg_i = Hg_i = (Hg_i)H$$ ← H is the identity element in the set of cosets

$$Hg_i Hg_i^{-1} = H$$

For a normal subgroup $H \triangleleft G$, the set of right cosets forms a group - Quotient group G/H

$$[g_i] = Hg_i$$



Example: $G = \{ na\hat{x} \mid n \in \mathbb{Z} \}$ a - dimensionful lattice constant

$H = \{ 3na\hat{x} \mid n \in \mathbb{Z} \}$

Cosets: $H = \{ 0\hat{x}, \pm 3a\hat{x}, \pm 6a\hat{x}, \ldots \}$

$H + a\hat{x} = \{ a\hat{x}, -2a\hat{x}, 4a\hat{x}, -5a\hat{x}, 7a\hat{x}, \ldots \}$

$H + 2a\hat{x} = \{ 2a\hat{x}, -a\hat{x}, 5a\hat{x}, -4a\hat{x}, 8a\hat{x}, \ldots \}$

$H \triangleleft G$ $\quad na\hat{x} + H = H + na\hat{x}$

$$G = H \cup (H + a\hat{x}) \cup (H + 2a\hat{x}) \qquad |G:H| = 3$$

$$(H) + (H + a\hat{x}) = H + a\hat{x} \qquad H + H = H$$

$$H + (H + 2a\hat{x}) = H + 2a\hat{x}$$

$$(H + a\hat{x}) + (H + a\hat{x}) = H + 2a\hat{x}$$

$$(H + a\hat{x}) + (H + 2a\hat{x}) = H + 3a\hat{x} = H$$

$$(H + 2a\hat{x}) + (H + 2a\hat{x}) = H + 4a\hat{x} = H + a\hat{x}$$

$$H \sim [0]$$

$$H + a\hat{x} \sim [1]$$

$H + 2a's \sim [2]$

$[0] + [1] = [1]$

$[0] + [2] = [2]$

$[1] + [2] = 0$

addition modulo 3

$G/H = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$

In Quantum mechanics

$\phi : G \longrightarrow K$

group of

unitary operaters on our QM

Symmetries Hilbert space

$$\phi : g \to \phi(g) \in K$$

Special subset of functions that are compatible with group multiplication

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$$ ← <u>group homomorphism</u>

E_G - identity in G

E_K - identity in K

$$\phi(E_G) = E_K$$

$$\phi(g^{-1}) = [\phi(g)]^{-1}$$

Example: let $\vec{L}$ be a vector of $\boxed{\text{spin } \ell}$ angular momentum generators

let $SO(3)$ be the group of 3D rotations

$(\hat{n}, \theta) \in SO(3)$

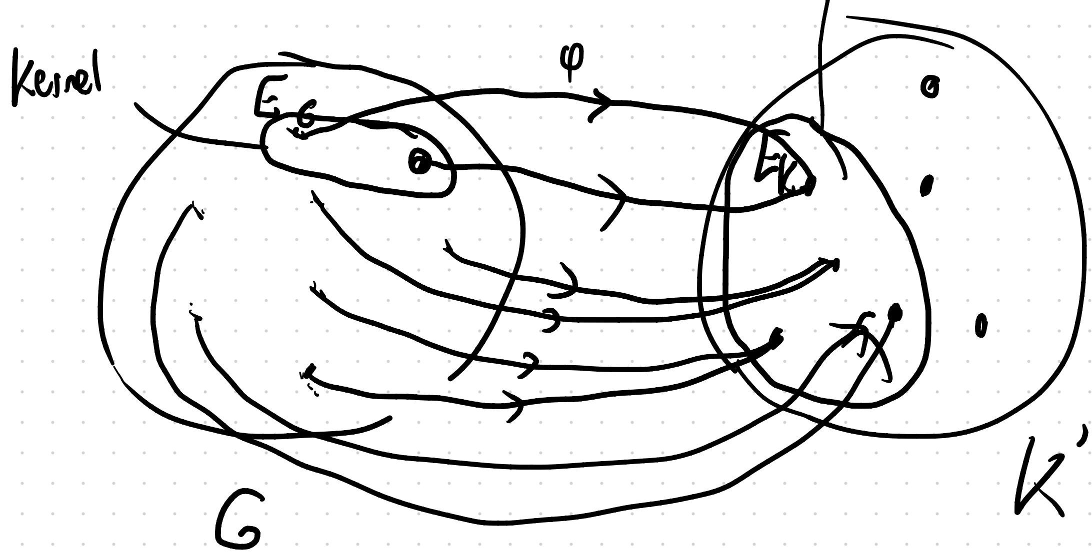
$\boxed{\varphi:}\ SO(3) \to U(2\ell+1)$

$(\hat{n}, \theta) \to e^{-i\hat{n}\cdot\vec{L}\theta/\hbar}$

Given $\varphi: G \to K$ a homomorphism

image $\mathrm{Im}(\varphi) = \{\varphi(g) \mid g \in G\} \subset K$

kernel $\mathrm{Ker}(\varphi) = \{g \mid g \in G,\ \varphi(g) = E_K\} \subset G$



(2) $\operatorname{Ker}(\varphi) \triangleleft G$ is a normal subgroup of G

(1) $\operatorname{Im}(\varphi) \subset K$ is a subgroup of K

pf Need

- $E_K \in \operatorname{Im}(\varphi) \rightarrow \varphi(E_G) = E_K$
- $k \in \operatorname{Im}(\varphi) \Rightarrow k^{-1} \in \operatorname{Im}(\varphi) \rightarrow$ if $k = \varphi(g)$
  $k^{-1} = [\varphi(g)]^{-1} = \varphi(g^{-1})$
- $k_1, k_2 \in \operatorname{Im}(\varphi) \Rightarrow k_1 k_2 \in \operatorname{Im}(\varphi)$

  $\hookrightarrow k_1 = \varphi(g_1)$

  $k_2 = \varphi(g_2)$

  $k_1 k_2 = \varphi(g_1)\varphi(g_2) = \varphi(g_1 g_2)$

Pf that $\operatorname{Ker}\varphi \triangleleft G$ $\qquad \operatorname{Ker}(\varphi) = \{ g \mid g \in G, \varphi(g) = E_K \}$

① $\varphi(E_G) = E_K \Rightarrow E_G \in \operatorname{Ker}(\varphi)$

② $g \in \operatorname{Ker}\varphi \Rightarrow \varphi(g) = E_K = [\varphi(g)]^{-1} = \varphi(g^{-1})$

$\Rightarrow g^{-1} \in \operatorname{Ker}\varphi$

③ $g_1, g_2 \in \operatorname{Ker}\varphi \quad \varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2) = E_K E_K = E_K$

$\Rightarrow g_1 g_2 \in \operatorname{Ker}\varphi$

Let $g \in \mathrm{Ker}\,\varphi$ $\quad g' \in G$

$$\varphi(g' g (g')^{-1}) = \varphi(g')\boxed{\varphi(g)}\varphi(g'^{-1})$$

$$= \varphi(g')\varphi(g'^{-1})$$

$$= \varphi(g')[\varphi(g')]^{-1} = E_K$$

$$g' g g'^{-1} \in \mathrm{Ker}\,\varphi$$

$$\Rightarrow g' \,\mathrm{Ker}\,\varphi\, (g')^{-1} = \mathrm{Ker}\,\rho$$

Putting it all together: <u>First Isomorphism Theorem:</u>

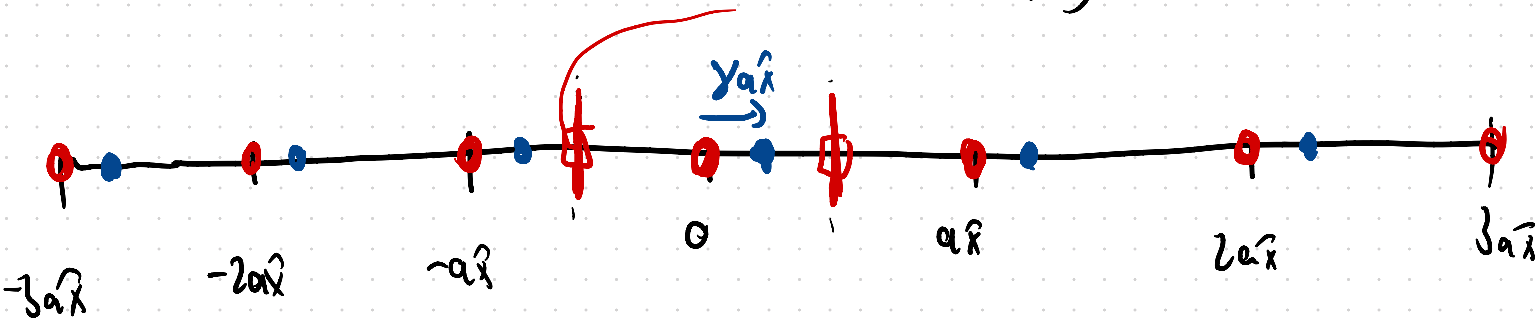
G, K groups, $\varphi: G \to K$ is a group homomorphism

$G/_{\ker\varphi} \simeq \mathrm{Im}\,\varphi$ $\qquad$ $\mathrm{Im}\,\varphi$ is in 1-to-1 correspondence with right cosets of $\ker\varphi$

Example: $\mathbb{R}^1 = \{\beta a\hat{x} \mid \beta \in \mathbb{R}\}$ the 1d translation group

$T = \{na\hat{x} \mid n \in \mathbb{Z}\}$

$$T \triangleleft \mathbb{R}^1 \qquad \mathbb{R}^1 = \bigcup_{y \in [-\frac{a}{2}, \frac{a}{2}]} \left(T + ya\hat{x}\right)$$



Claim: $\mathbb{R}^1/T = U(1)$ - which is the unit circle

pf: $\varphi(\beta a\hat{x}) = e^{2\pi i\beta} \in U(1)$

$$\varphi(\beta_1 a\hat{x}+\beta_2 a\hat{x}) = e^{2\pi i(\beta_1+\beta_2)} = e^{2\pi i\beta_1}e^{2\pi i\beta_2} = \varphi(\beta_1 a\hat{x})\varphi(\beta_2 a\hat{x})$$

$$\mathrm{Ker}(\varphi) = \{\eta a\hat{x} \mid e^{2\pi i\eta} = 1\} = \{n a\hat{x} \mid n \in \mathbb{Z}\} = T$$

$$\mathrm{Im}(\varphi) = U(1)$$

$$\Rightarrow \quad \mathbb{R}^1/T = U(1)$$

- unit cell of the Bravais lattice T