Lecture 3

Reminders: · HW 1 is now posted

· Due 2/6 on gradescope

· Office hours Mondays 4-5pm, Zoom link on course website

Last week: Groups, subgroups, cosets, quotient groups

→ 1st isomorphism theorem

$\phi: G \to K$ a homomorphism

- $\ker \varphi \triangleleft G$

$\mathrm{Im}\,\varphi \subset K$

$G/_{\ker\varphi} \cong \operatorname{Im}\varphi$

isomorphic: bijective homomorphism

One last point about quotient groups

$H \triangleleft G$, H is a normal subgroup of G

$G = H \cup Hg_1 \cup Hg_2 \cdots \cup Hg_{n-1}$

quotient group $G/_H = \{H, Hg_1, Hg_2, \ldots Hg_n\}$

In <u>some</u> cases, there exists a <u>homomorphism</u>

$i: G/H \to G$

$i(Hg_i) = g_i \in G$

$i(H) = E \in G$

<u>If</u> i exists and is a group homomorphism, then

$\operatorname{Im} i = K = \{E, g_1, g_2, \ldots g_n\} \subset G$ is a subgroup of

G isomorphic to G/H

<u>and</u> any $g \in G$ there exists a unique $h \in H$ and $k \in K$

s.t. $g = hk$ ← coset decomposition

$$\Rightarrow \quad G = HK$$

when this is possible, we say G is a semidirect product $G = H \rtimes K$ of H with K

Example: The group of rigid transformations of 3D space ← Euclidean group $\mathbb{E}(3)$

- rotations
- reflections
- translations

$$\mathbb{E}(3) \ni g = \{R \mid \vec{v}\}$$ - seitz symbol for g

$R \in O(3)$ rotation or reflection

$\vec{v} \in \mathbb{R}^3$ translations

Action on points in space

$$g\vec{x} = R\vec{x} + \vec{v}$$

Multiplication: $g_1 = \{R_1 \mid \vec{v}_1\}$

$g_2 = \{R_2 \mid \vec{v}_2\}$

$$(g_1 g_2)\vec{x} = g_1(g_2\vec{x}) = g_1(R_2\vec{x} + \vec{v}_2)$$

$$= R_1(R_2\vec{x}+\vec{v}_2)+\vec{v}_1$$

$$= R_1R_2\vec{x}+(\vec{v}_1+R_1\vec{v}_2)$$

$$= \{R_1R_2 \mid R_1\vec{v}_2+\vec{v}_1\}$$

$$\Rightarrow \boxed{\{R_1 \mid \vec{v}_1\}\{R_2 \mid \vec{v}_2\} = \{R_1R_2 \mid R_1\vec{v}_2+\vec{v}_1\} \quad \text{Multiplication rule for Seitz symbols}}$$

$$g = \{R \mid \vec{v}\} \rightarrow g^{-1} = \{R^{-1} \mid -R^{-1}\vec{v}\}$$

$$gg^{-1} = \{R|\vec{v}\}\{R^{-1}|-R^{-1}\vec{v}\}$$

$$= \{RR^{-1}|-RR^{-1}\vec{v}+\vec{v}\} = \{E|\vec{0}\} \leftarrow \text{identity transformation}$$

the group of translations $\{\{E|\vec{v}\} \mid \vec{v} \in \mathbb{R}^3\} \cong \mathbb{R}^3 \subset \mathbb{E}(3)$

is a normal subgroup of $\mathbb{E}(3)$

Check $\{R|\vec{d}\}\{E|\vec{v}\}\{R|\vec{d}\}^{-1}$

$$= \{R|\vec{d}+R\vec{v}\}\{R^{-1}|-R^{-1}\vec{d}\}$$

$$= \{E|R\vec{v}\} \in \mathbb{R}^3$$

$$\mathbb{R}^3 \triangleleft E(3)$$

$$\{R|\vec{v}\} = \{E|\vec{v}\}\{R|\vec{0}\}$$

$$E(3) = (\mathbb{R}^3)[O(3)]$$

$$E(3) = \mathbb{R}^3 \rtimes O(3)$$

Direct product: $G \times H = \{(g,h) \mid g \in G, h \in H\}$

$$(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 h_2)$$

$$(g,h) = (g,E_H)(E_G,h)$$
$$= GH$$

Semidirect product: $(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 \varphi_{g_1}(h_2))$

φ_g: an action of G on H

How do we use groups in solid state physics

$$H = \frac{p^2}{2m} + V(x) \qquad \vec{x} \to \vec{x}' = g\vec{x} \qquad g \in G$$

$$[x, p] = i\hbar \qquad \vec{p} \to \vec{p}' = g^{-1}\vec{p}$$

$$\underbrace{\psi'(x) \to \psi(g^{-1}x)}_{\text{all } g \text{ s.t. } H' = H}$$

look for unitary operators U_g for each $g \in G$
that implement these transformations

$$U_g^\dagger \vec{X} U_g = X' \qquad \text{want } U_{g_1} U_{g_2} = U_{g_1 g_2}$$

$$U_g^\dagger \vec{P} U_g = P'$$

$$|\psi'\rangle = U_g |\psi\rangle$$

$\Rightarrow$ $\varphi : g \to U_g$ a homomorphism

$G \to U(V)$ $\quad$ V - our QM hilbert space

$U(V)$ - group of unitary operators on

V

Def a (unitary) representation of a group G is:

- a vector space V
- a homomorphism $\rho : G \to U(V)$

our symmetry group

group of unitary operators/matrices

$\rho(g) \in U(V)$ is the representative of G

Example: representations of $\mathbb{R}^3$ in QM

$\mathbb{R}^3 \ni \vec{v} \to U_{\vec{v}} = e^{-\frac{i}{\hbar}\vec{p}\cdot\vec{v}}$

$$U_{\vec{v}_1} U_{v_2} = e^{-\frac{i}{\hbar}p\cdot v_1} e^{-\frac{i}{\hbar}p\cdot v_2} = e^{-\frac{i}{\hbar}\vec{p}\cdot(v_1+v_2)} = U_{v_1+v_2}$$

Example: $SU(2)$ $(\hat{n}, \theta)$ $\theta \in [0, 2\pi)$

a unit vector

$(\hat{n}, \theta) \to \cos\frac{\theta}{2}\sigma_0 + i\sin\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}$ unitary 2x2 matrix

σ_0 - 2x2 identity matrix

$\vec{\sigma}$ - a vecter of Pauli matrices

defining representation

$\ell = 1$

$$L_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$L_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$L_z = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix}$$

$$(\hat{n}, \theta) \rightarrow e^{-i \hat{n} \cdot \vec{L} \theta}$$

SU(2)

3x3 unitary matrix

angular momentum ℓ representation of $SU(2)$

is a representation $SU(2) \to U(2\ell+1)$

$$\boxed{SO(3) = SU(2)/\{E, -E\}}$$

let G be a group, ρ be a representation of G

on a vector space V $[\rho : G \to U(V)]$

$|v\rangle \in V$

$\rho(g)|v\rangle \in V$

we can look for subsets $W \subset V$ s.t.

for all $|w\rangle \in W$, $\rho(g)|w\rangle \in W$ for all g

- <u>invariant subspaces of ρ</u>

Since $\rho(g)$ are all unitary if W is an invariant subspace, then so is

$$W^{\perp} = \{ |v\rangle \in V \mid \langle v|w\rangle = 0 \text{ for all } |w\rangle \in W \}$$

proof: W is invariant $\Rightarrow$ $|w\rangle \in W$

$$\rho(g)|w\rangle \in W$$

let $|v\rangle \in W^\perp$ $\quad \langle v|\rho(g)|w\rangle = 0$

$$\rightarrow \langle w|\rho^\dagger(g)|v\rangle = 0$$

$$\Rightarrow \langle w|\rho(g^{-1})|v\rangle = 0 \qquad \rho^\dagger(g) = \rho(g^{-1})$$

$$\Rightarrow \rho(g^{-1})|v\rangle \in W^\perp$$

$\Rightarrow$ $W^\perp$ is an invariant subspace

$$V = W \oplus W^{\perp}$$

as a matrix

$$\rho(g) = \begin{pmatrix} \rho_{11}(g) & \rho_{12}(g) & \rho_{13}(g) & \cdots \\ \rho_{21}(g) & \rho_{22}(g) & \cdots & \\ \vdots & & \ddots & \end{pmatrix}$$

in a general basis

we can pick a new basis

$$\Big\{ \overbrace{|w_1\rangle, |w_2\rangle \ldots |w_N\rangle}^{\text{basis for } W}, \underbrace{|v_1\rangle, |v_2\rangle \ldots |v_M\rangle}_{\text{basis for } W^{\perp}} \Big\}$$

in this new basis

$$\boxed{\rho(g)} = W \begin{array}{c} \begin{array}{cc} W & W^{\perp} \end{array} \\ \left(\begin{array}{c|c} \langle w_i|\rho(g)|w_j\rangle & \langle w_i|\rho(g)|v_j\rangle \\ \hline \langle v_i|\rho(g)|w_j\rangle & \langle v_i|\rho(g)|v_j\rangle \end{array}\right) \end{array} = \left(\begin{array}{c|c} \rho_W(g) & 0 \\ \hline 0 & \rho_{W^{\perp}}(g) \end{array}\right)$$

for <u>every</u> group element

$$\rho(g) = \rho_W(g) \oplus \rho_{W^{\perp}}(g)$$

we say that ρ is <u>reducible</u>

if a representation is not reducible, then it is an irreducible representation (irrep)

has only $\{0\}$ and V as invariant subspaces

$$\boxed{U_g H U_g^{\dagger} = H}$$