Lecture 4

Reminder: · HW 1 posted, Due 2/6

· Office hours Mondays 4-5

Recap: Representation ρ of a group G on a vector space V

homomorphism $\rho : G \to U(V)$

group of unitary operators on V

$g \to \rho(g) \in U(V)$

Invariant subspace: $W \subset V$ s.t. $\rho(g) W \subset W$

if a representation has an invariant subspace then we can choose a basis where for every g,

$$\rho(g) = \left(\begin{array}{c|c} \rho_W(g) & 0 \\ \hline 0 & \rho_{W^\perp}(g) \end{array}\right)$$

equivalent up to a change of basis, "unitarily equivalent"

$$\rho \cong \rho_W \oplus \rho_{W^\perp}$$

$$V = W \oplus W^\perp$$

ρ is <u>reducible</u>

if ρ has no invariant subspaces (other than $\{0\}$ and V) then we say ρ is <u>irreducible</u>

Example: let G be any group I can always construct a special 1D irrep $V = \mathbb{C}$

$$U(V) = \{e^{i\phi},\ \phi \in [0, 2\pi)\}$$

$$\rho : g \to e^{i\cdot 0} = 1 = \rho(g) \quad \text{- trivial representation}$$

Example: consider two spin-$\frac{1}{2}$ particles

$$V = \{|\uparrow\uparrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\downarrow\rangle\}$$

under a rotation $(\hat{n}, \theta)$ each particle transforms

$$\rho_{\frac{1}{2}}((\hat{n}, \theta)) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}}$$

the set of 2 spin-$\frac{1}{2}$ particles transforms in a representation

$$\rho((\hat{n}, \theta)) = \rho_{\frac{1}{2}}((\hat{n}, \theta)) \otimes \rho_{\frac{1}{2}}((\hat{n}, \theta))$$

there is a nontrivial invariant subspace

$$W = \left\{ \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \right) \right\}$$ – spin 0 subspace

$$W^{\perp} = \left\{ |\uparrow\uparrow\rangle, \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \right), |\downarrow\downarrow\rangle \right\}$$ – spin-1 subspace

$$\rho((\hat{n},\theta)) = \begin{array}{c c} & \begin{array}{c c} W & W^{\perp} \end{array} \\ \begin{array}{c} W \\ W^{\perp} \end{array} & \left(\begin{array}{c|c c c} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & e^{-i\hat{n}\cdot\vec{L}\theta} & \\ 0 & & & \end{array} \right) \end{array}$$

$\vec{L}$ – spin-1 matrices

Clebsch-Gordan coefficients – bases for the invariant subspaces

Schur's Lemma (Part I)

Consider a group G and two irreducible representations

$$\rho_1 : G \to U(V_1)$$
$$\rho_2 : G \to U(V_2)$$

$v_1 \in V_1$, $v_2 \in V_2$

If we have a matrix $H : V_1 \to V_2$ $(Hv_1 = v_2)$

such that $\boxed{H\rho_1(g) = \rho_2(g)H}$ for all $g \in G$

then either: ① $H = 0$ ✓ or

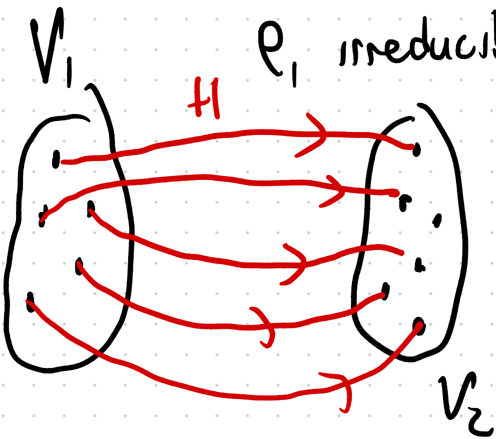
② H is invertible

Proof: lets look at $\operatorname{Ker} H = \{v \in V_1 \mid Hv = 0\}$

if $v \in \ker H$, consider $\rho_1(g)v$

$$H\rho_1(g)\vec{v} = \rho_2(g)\underline{H\vec{v}} = 0$$

$\Rightarrow \rho_1(g)\vec{v} \in \ker H \longrightarrow \ker H$ is an invariant subspace of ρ_1



ρ_1 irreducible $\Rightarrow$ $\ker H = V_1 \longrightarrow H = 0$ is the zero matrix

or

$\ker H = \{\vec{0}\} \rightarrow Hv_1 = Hv_2$

H is one-to-one $\Rightarrow H(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 \in \ker H$

$\Rightarrow v_1 = v_2$

Now lets look at $\mathrm{Im}\, H = \{ w \in V_2 \mid w = Hv_1 \text{ for some } v_1 \in V_1 \}$

if $w \in \operatorname{Im} H \qquad w = H v_1$

$\rho_2(g) w = \rho_2(g) H v_1 = H \rho_1(g) v_1 \Rightarrow \rho_2(g) w \in \operatorname{Im} H$

$\Rightarrow \operatorname{Im} H$ is an invariant subspace of ρ_2

ρ_2 irreducible $\Rightarrow \operatorname{Im} H = \begin{cases} \vec{0} \leftarrow & H = 0 \\ \underline{V_2} \leftarrow & H \text{ is surjective "onto"} \end{cases}$

$\Rightarrow \quad H = 0$

H is one-to-one and surjective $\rightarrow$ invertible

<u>Part 2</u>: Consider a group G one irreducible representation $\rho_1 = \rho_2 = \rho$ on a finite-dimensional vector space V_1

$H: V_1 \to V_1 \qquad H\rho_1(g) = \rho_1(g)H \quad ([H, \rho(g)] = 0)$

then either $\begin{cases} H = 0 \\ H \text{ is invertible} \longrightarrow \lambda\, \underset{\uparrow \text{identity}}{Id} \end{cases} \quad \lambda \in \mathbb{C}$

Pf: suppose H is invertible. Since H is a finite-dim. square matrix it has at least one eigenvector $\vec{v}$ and eigenvalue λ

Consider $B = H - \lambda\, Id$

$[\rho(g), B] \Rightarrow$ B is either invertible or 0 by Schur's lemma pt 1

but $B\vec{v} = 0 \Rightarrow \ker B \neq \{\vec{0}\} \Rightarrow$ B is not invertible

$B=0 \Rightarrow H = \lambda \, Id$

This applies to QM: let G be the symmetry group of our Hamiltonian H

$\{|\psi_i\rangle, \ i=1,\dots N\}$ transform in an irrep ρ of G

$$U_g|\psi_i\rangle = \sum_j |\psi_j\rangle \rho_{ji}(g)$$ s.t. $\rho_{ji}(g)$ is an irrep

$$\boxed{U_g^\dagger H U_g = H}$$

$[H]_{ij} = \langle \psi_i | H | \psi_j \rangle$ - matrix elements of the Hamiltonian

$$[H_{ij}] = \langle \underbrace{\psi_i | U_g^\dagger}\, H \left[U_g | \psi_j \rangle \right]$$

$$= \sum_{kl} \rho_{ik}^\dagger(g)\, [H]_{kl}\, \rho_{lj}(g)$$

$$[H] = \rho^\dagger(g) \boxed{[H]} \rho(g) \Rightarrow \left[[H], \rho(g) \right] = 0$$

$\Rightarrow$ Schur's lemma $\Rightarrow$ $[H]_{ij} = \delta_{ij} E_n$

$\rightarrow$ $\boxed{\text{states transforming in an irrep of the symmetry group are degenerate}}$

Example: Non-relativistic hydrogen atom

$\{|n\ell m_z\rangle \mid m_z = -\ell, \dots, \ell\}$ transform in the spin-ℓ representation of the rotation group

$$\langle n\ell m_z | H | n\ell m_z' \rangle = E_{n\ell}\, \delta_{m_z, m_z'}$$

energies are independent of m_z

Part 2.5 of Schur's lemma: G a group

$\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$

finite dimensional, irreducible

$H: V_1 \to V_2 \qquad \rho_2(g) H = H \rho_1(g)$

If H is invertible, then $\rho_1 \cong \rho_2$ ρ_1 is unitarily equivalent to ρ_2

Pf.: $\boxed{\rho_2(g) H = H \rho_1(g)}$ $\dagger$

$H^\dagger : V_2 \to V_1$ (b/c of the transpose)

$\boxed{H^\dagger \rho_2^\dagger(g) = \rho_1^\dagger(g) H^\dagger}$

$\Downarrow$

$H^\dagger \rho_2(g^{-1}) = \rho_1(g^{-1}) H^\dagger \to H^\dagger$ satisfies Schur's lemma

Consider $H^\dagger H : V_1 \to V_1$

$[H^\dagger H, \rho_1(g)] = 0 \Rightarrow$ Schur's Lemma 2 $\Rightarrow H^\dagger H = \lambda\, Id$

H invertible $\quad H^\dagger = \lambda H^{-1}$

$$U = \frac{1}{\sqrt{\lambda}} H$$

$$U^\dagger = \frac{1}{\sqrt{\lambda}} H^\dagger = \sqrt{\lambda} H^{-1} = U^{-1}$$

$\Rightarrow \quad H\rho_1(g) = \rho_2(g) H$

$\Rightarrow \rho_1(g) = H^{-1}\rho_2(g) H$

$= U^\dagger \rho_2(g) U \Rightarrow \rho_1 \cong \rho_2$