Lecture 5

Recap: Schur's Lemma: if we have a group G and two irreducible representations

$$\rho_1 : G \to U(V_1)$$
$$\rho_2 : G \to U(V_2)$$

and if we have a linear map $H : V_1 \to V_2$ such that $H\rho_1(g) = \rho_2(g)H \quad \forall g \in G$

then ① $H = 0$

② H is invertible $\Rightarrow$ $\rho_1 \cong \rho_2$ $\Rightarrow$ there is a basis where $\rho_1 = \rho_2$ and $H = \lambda\, Id$

Character Theory

Let us:

- tell when two representations are the same

- tell when a representation is irreducible
- count/enumerate irreducible representation

group G, and a representation ρ

The <u>character</u> χ_ρ of the representation ρ is

a function $\chi_\rho : G \to \mathbb{C}$

$$\chi_\rho(g) = tr[\rho(g)]$$

1. if two representations ρ_1 and ρ_2 are unitarily

equivalent $\quad \rho_1 \cong \rho_2 \Rightarrow U\rho_1(g)U^\dagger = \rho_2(g) \ \forall g$

U is a unitary matrix

$$\chi_{\rho_2}(g) = tr[\rho_2(g)] = tr[U\rho_1(g)U^\dagger]$$
$$= tr[\rho_1(g)] = \chi_{\rho_1}(g)$$

$$\rho_1 \cong \rho_2 \Rightarrow \chi_{\rho_1} = \chi_{\rho_2}$$

(2) if $g_2 = g g_1 g^{-1}$ then

$$\chi_\rho(g_2) = tr[\rho(g_2)] = tr[\rho(g g_1 g^{-1})]$$

$$= \mathrm{tr}\left[\rho(g)\rho(g_1)\rho(g^{-1})\right]$$

$$= \mathrm{tr}\left[\rho(g^{-1})\rho(g)\rho(g_1)\right]$$

$$= \mathrm{tr}\left[\rho(g_1)\right]$$

$$= \chi_\rho(g_1)$$

$\Rightarrow$ Characters are constant on conjugacy classes

(g_1, g_2 are in the same conjugacy class, $\chi_\rho(g_1) = \chi_\rho(g_2)$ for all representations)

3 If ρ_1 and ρ_2 are representations,

direct sum representation $\rightarrow \rho = \rho_1 \oplus \rho_2$

$$\rho(g) = \left(\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right)$$

$$\rho : G \rightarrow V_1 \oplus V_2$$

$$\mathrm{tr}\, \rho(g) = \mathrm{tr}\, \rho_1(g) + \mathrm{tr}\, \rho_2(g)$$

$$\underline{\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2}}$$

Using Schur's Lemma we will prove the "Wonderful Orthogonality Theorem"

Lets take a finite group G and two irreducible representations $\rho_1 : G \to U(V_1)$

$\rho_2 : G \to U(V_2)$

and <u>any</u> linear map $A : V_1 \to V_2$ — $\dim V_2 \times \dim V_1$ matrix

lets form $$A_G = \sum_{g \in G} \rho_2(g^{-1}) A \rho_1(g)$$

Claim: $A_G \rho_1(g) = \rho_2(g) A_G$

pf: $$A_G \rho_1(g) = \left[\sum_{g' \in G} \rho_2(g'^{-1}) A \rho_1(g')\right] \rho_1(g)$$

$$= \sum_{\boxed{g' \in G}} \rho_2(g'^{-1}) A \rho_1(g'g)$$

let $g'' = g'g$

$\downarrow$

$g' = g'' g^{-1}$

$$= \sum_{g'' \in G} \rho_2(g g''^{-1}) A \rho_1(g'')$$

$g'^{-1} = g g''^{-1}$

$$= \sum_{g'' \in G} \rho_2(g) \rho_2(g''^{-1}) A \rho_1(g'')$$

$$= \rho_2(g) A_G$$

so A_G satisfies the conditions of Schur's lemma

$\Rightarrow$ Either $\rho_1 \neq \rho_2 \Rightarrow A_G = 0$

or

$\rho_1 \cong \rho_2 \Rightarrow$ in a basis where $\rho_1 = \rho_2$ $A_G = \lambda \, \mathrm{Id}$

Lets apply this to a special choice

$$A = E_{ij} \qquad \begin{array}{cc} & \xrightarrow{\qquad\qquad} j\text{th} \\ i\text{th}\downarrow & \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & 1 & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} \end{array}$$

$$[E_{ij}]^{\mu\nu} = \delta_{i\mu}\delta_{j\nu}$$

$$[E_{ij}]_G^{\mu\nu} = \sum_{g\in G} \left[\rho_2(g^{-1}) E_{ij}\, \rho_1(g)\right]^{\mu\nu}$$

$$= \sum_{g\in G}\sum_{\alpha\beta} [\rho_2(g^{-1})]^{\mu\alpha}[E_{ij}]^{\alpha\beta}[\rho_1(g)]^{\beta\nu}$$

$$= \sum_{g\in G}\sum_{\alpha\beta} [\rho_2(g^{-1})]^{\mu\alpha}\delta_{i\alpha}\delta_{j\beta}[\rho_1(g)]^{\beta\nu}$$

$$= \sum_{g \in G} [\rho_2(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\simeq \rho_2 \\ \lambda \delta_{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}$$

to find λ, let $\underbrace{\rho_1 = \rho_2}$ and take

$$\mathrm{tr}\left[E_{ij}\right]_G = \mathrm{tr}\left[\sum_{g \in G} [\rho_1(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu}\right]$$

$$= \sum_{\mu} \sum_{g \in G} [\rho_1(g^{-1})]^{\mu i} [\rho_1(g)]^{j\mu}$$

$$= \sum_{g \in G} [\rho_1(g)\rho_1(g^{-1})]^{ji} = \sum_{g \in G} \delta_{ij}$$

$$= |G|\delta_{ij}$$

$$|G|\delta_{ij} = tr(\lambda \delta_{\mu\nu}) = \lambda \dim \rho_1$$

$$\Rightarrow \lambda = \frac{\delta_{ij}|G|}{\dim \rho_1}$$

If ρ_1 and ρ_2 are irreducible representations then

$$\boxed{\sum_{g\in G} [\rho_2(g^{-1})]^{\mu i} [\rho_1(g)]^{j\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\dim \rho_1}\delta_{ij}\delta_{\mu\nu} & \text{if } \rho_1 = \rho_2 \end{cases}}$$

(if $\dim V_1 \neq \dim V_2 \Rightarrow \rho_1 \not\cong \rho_2$)

trying this to characters:

$$\sum_{g\in G} \sum_{\mu\nu} [\rho_2(g^{-1})]^{\mu\mu} [\rho_1(g)]^{\nu\nu} = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ \frac{|G|}{\cancel{\dim \rho_1}} \cancel{\dim \rho_1} & \rho_1 \cong \rho_2 \end{cases}$$

$$\parallel$$

$$\sum_{g\in G} \chi_{\rho_2}(g^{-1}) \chi_{\rho_1}(g)$$

$$\rho_2(g^{-1}) = [\rho_2(g)]^{-1}$$
$$= [\rho_2(g)]^{\dagger}$$

for ρ_1 and ρ_2 irreducible

$$\chi_{\rho_2}(g^{-1}) = \mathrm{tr}(\rho_2(g^{-1})) = \mathrm{tr}[(\rho_2(g))^{\dagger}]$$

$$\boxed{\frac{1}{|G|}\sum_{g\in G}\left(\chi_{\rho_2}(g)\right)^{*}\chi_{\rho_1}(g) = \begin{cases} 0 & \text{if } \rho_1 \not\cong \rho_2 \\ 1 & \text{if } \rho_1 \cong \rho_2 \end{cases}}$$

$$= \left(\mathrm{tr}\left[\rho_2(g)\right]\right)^{*}$$

$$= \left[\chi_{\rho_2}(g)\right]^{*}$$

$$\| $$

$$\langle \chi_{\rho_2}, \chi_{\rho_1} \rangle$$

Characters of irreducible representations are orthonormal under this inner product

(Peter-Weyl theorem for finite groups)

lets say we have a reducible representation

$$\rho = \rho_1 \oplus \rho_2 \oplus \rho_2 \oplus \rho_j \oplus \dots$$

ρ_i are irreducible

$$\chi_\rho = \chi_{\rho_1} + \chi_{\rho_2} + \chi_{\rho_2} + \chi_{\rho_3} + \dots$$

$$\rho = \bigoplus_{i=1}^{N} n_i \rho_i = \underbrace{\rho_1 \oplus \rho_1 \oplus \dots \rho_1}_{n_1} \oplus \underbrace{\rho_2 \oplus \rho_2 \oplus \dots \rho_2}_{n_2} \oplus \dots$$

$$\chi_\rho = \sum_{i=1}^{N} n_i \chi_{\rho_i}$$

n_i = multiplicity of ρ_i in the decomposition of ρ

$$\langle \chi_{\rho_j}, \chi_\rho \rangle = \sum_{i=1}^{N} \langle \chi_{\rho_j}, \chi_{\rho_i} \rangle n_i = n_j$$

Example: The group D_2 from HW1

$$D_2 = \{E, C_{2x}, C_{2y}, C_{2z}\}$$

$$C_{2x}^2 = C_{2y}^2 = C_{2z}^2 = E$$

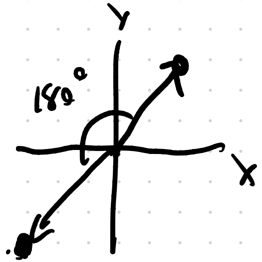
$$C_{2i} C_{2j} = C_{2j} C_{2i} \text{ for } i,j = x,y,z$$

Defining representation (vector representation)

$$\mathbb{R}^3 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\rho_V(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \rho_V(C_{2x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \rho_V(C_{2y}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\rho_V(C_{2z}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Invariant subspaces $\left\{\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}\right\}, \left\{\begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}\right\}$

Abelian groups (groups where all elements commute) all irreducible representations are 1D

Character table

	E	C_{2x}	C_{2y}	C_{2z}
A	1	1	1	1
B_1	1	1	-1	-1
B_2	1	-1	1	-1
B_3	1	-1	-1	1

$$\rho_V = \rho_{B_1} \oplus \rho_{B_2} \oplus \rho_{B_3}$$