Problem 1 *The exponential function* 10 points
In Lecture Note #4, we argued *incompletely* that the only function whose derivative is itself is the exponential function. For this problem provide the complete argument. In particular:

(a) Find a function $f(x)$ such that $f'(x) = f(x)$ (fill in the steps skipped in the notes; you can assume the Taylor expansion there is convergent).
(b) Then show that upto a constant multiplier (and barring specially constructed cases), the function found in part (a) is the only such function whose derivative is itself.

Hint: Complete the ideas outlined in Lecture Note #4 by filling in the blanks and all the skipped steps.

Problem 2 *Orthogonality I* 20 points
In class, we showed that the functions $f_n(t) := e^{int}$ for $n \in \mathbb{Z}$ are orthogonal under the inner product defined as

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt$$

where $\overline{g(t)}$ is the complex conjugate of $g(t)$. Use this result to prove the following relations under the *new* inner product:

$$\langle f, g \rangle := \frac{1}{2\pi} \int_0^{2\pi} f(t)g(t)dt$$

where $m, n \in \mathbb{Z}$ and $m^2 \neq n^2$.

(a) Orthogonality of $f_n(t) := \sin(nt)$ and $g_m(t) := \cos(mt)$.
(b) Orthogonality of the set $\{f_n(t)\}_n$
(c) Orthogonality of the set $\{g_n(t)\}_n$
(d) $\langle \sqrt{2\pi} f_n, \sqrt{2\pi} f_n \rangle = \langle \sqrt{2\pi} g_n, \sqrt{2\pi} g_n \rangle = \pi$ for $n \neq 0$.

Problem 3 *Pearson correlation* 10 points
In Lecture Note #4 we showed that the Pearson correlation has to be between -1 and 1 by clumsily switching to *population* statistics rather than sticking to *sample statistics*. For this problem, show that essentially the same argument carries through and we didn't have to make the switch.

Problem 4 *SNR & Decibels* 15 points

(a) If a signal is measured as 2.5 V and the noise is 28 mV, what is the SNR in dB?
(b) Suppose a sinusoidal signal has been corrupted by a large amount of noise. If the RMS of the noise is 0.5 V and the SNR is 10 dB what is the RMS amplitude of the sinusoid?
(c) The attached file `signal_noise.mat` contains a variable x that has a 1 V sinusoidal signal "buried in noise" (noise is larger than signal). What is the SNR for this signal and noise?

Problem 5 *Ensemble averaging* 15 points
Load the data in `ensemble_data.mat`, which contains a data matrix. The data matrix contains 100 responses of a signal in noise. In this matrix, each row is a separate response.

(a) Plot several randomly selected samples of these responses. Is it possible to identify the signal from any single record?
(b) Construct and plot the ensemble average for these data.
(c) Construct and plot the ensemble standard deviation.

Problem 6 *Orthogonality II*   10 points
The file `two_var.mat` contains two variables `x` and `y`. Is either of these variables random? Are they orthogonal to each other?
Hint: Regarding randomness, think *autocorrelation*: lagged correlations of a function with itself.

Problem 7 *Cross correlation*   10 points
Consider two signals $y_1(t) = \cos(10t + 20)$ and $y_2 = \sin(10t - 10)$ where the phase is in degrees. Use cross-correlation to find the time delay in seconds between the two signals. Note that you will have to choose the sampling frequency and number of data points yourself.

Problem 8 *Effect of memory*   10 points
Construct an array, `x`, of Gaussian random numbers ($N = 2000$). Construct a new signal from this array where each data point is a five- point running average of `x` (i.e., `y(1) = mean(x(1:5));` `y(2) = mean(x(2:6)); ... y(N-5) = mean(x(N-5:N)))`. Plot the autocorrelation functions of both the Gaussian and averaged signals. Expand the lag axis to `-12` lags to observe the effect of memory on the auto-correlation function.

Problem 9 **Bonus:** *Roots of unity & an important matrix (hard)*   15 points
A complex number z is called a *primitive* n-th root of unity if n is the smallest positive integer with $z^n = 1$. For example, ± 1 and $\pm i$ are *fourth* roots of unity, but only $\pm i$ are primitive, while ± 1 are not primitive (since $1^n = 1$ and $(-1)^n = 1$ when $n = 1 \neq 4$ and $n = 2 \neq 4$ respectively).

Let ω be a primitive n-th root of unity. Find the inverse of the $n \times n$ matrix given by:

$$F_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{2(n-1)} \\ \vdots & \vdots & & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)^2} \end{bmatrix}$$

Hint: Start by considering the first few cases, say the $n = 2$ and $n = 3$ (skip $n = 1$, that might not be so insightful). Use MATLAB or CAS to find their inverses and then try to generalize the patterns you see. To confirm your claim, it might help to write a function for the (l, m) entry of F_n, i.e. $F_n(l, m, \omega) = \dots$ and then use the definition of matrix multiplication:

$$C = AB \implies c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

For the very last step, you might want to rearrange $z^n = 1$ and divide by $z - 1$.