Homework on Counting

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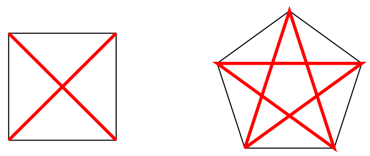
Problem 1. Given a regular n-gon,[1] a *diagonal* or *chord* is a line segment connecting two non-adjacent vertices. For example, all the possible diagonals of a regular square (4-gon) and a regular pentagon (5-gon) are shown in red below.



What is the total number of possible diagonals in a regular n-gon?

[1] i.e., a polygon with n sides

Problem 2. Count the total number of possible undirected graphs on n vertices. (Here we are ***not*** considering two graphs to be the same if they are isomorphic).

Problem 3 (Counting two different ways)**.** In this exercise we will establish some combinatorial identities by counting something two different ways.

a) We established that the number of paths from $(0,0)$ to $(k, n-k)$ on an integer grid where we only walk rightwards or upwards is $\binom{n}{k}$. Explain how to establish that the number of such paths can also be counted as $\binom{n-1}{k} + \binom{n-1}{k-1}$ to prove "Pascal's formula": $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. [2]

b) From n contestants, suppose k finalists are picked, out of which ℓ actually win. Explain how to count the ways for this to happen in two different ways to establish the identity $\binom{n}{k}\binom{k}{\ell} = \binom{n}{\ell}\binom{n-\ell}{k-\ell}$.

[2] Hint: each of the valid paths from $(0,0)$ to $(k, n-k)$ have to end in one of two ways.

Problem 4. Consider an election in which thirty people vote for three candidates.

a) How many voting outcomes are there, if we only care about how many votes each candidate gets?

b) How many voting outcomes are there, if we care about which candidate each voter voted for?

c) How many ways can we get a three-way tie between all three candidates (i.e., each candidate receives exactly one-third of the votes)?

Problem 5. Compute the number of *positive* integer solutions[3] to the equation $\sum_{i=1}^{n} x_i = k$, where k is some integer constant.[4]

[3] i.e., $x_i > 0$ for all i

[4] You may assume that $k \geq n$.