Homework on Pigeon Hole Principle and Principle of Inclusion-Exclusion

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Problem 1. Let $a_1, a_2, \ldots a_{52}$ be 52 integers. Prove that there must be two numbers a_i and a_j ($i \neq j$) in the list such that either $a_i + a_j$ or $a_i - a_j$ is a multiple of 100. *Hint:* Can you generalize the solution of Problem 1 in the worksheet to solve this?

Problem 2. Consider a square of side length s. Suppose we place 5 points on the square; the points either on the boundary of the square or inside. Prove that no matter how the points are placed, there are at least 2 points that are at most distance $s/\sqrt{2}$ apart. *Hint:* Divide the square into 4 equal squares of side length $s/2$. How far can any two points in the smaller square be?

Problem 3. A *sevens and zeros* number is a natural number whose decimal representation consists of one or more 7s followed by one or more 0s. Examples include $700, 770, 770000$, while non-examples include $123, 707, 77$. A *sevens* number is a natural number whose decimal representation consists of one or more 7s. Thus, they are one of $7, 77, 777, \ldots$

(a) Prove that for any non-zero natural number n, there is a sevens and zeros number k such that $n | k$. *Hint:* Consider the numbers $7, 77, 777, 7777, \ldots$.

(b) Conclude that if n is not divisible by 2 or 5 then n divides a sevens number.

Problem 4. Prove the principle of inclusion-exclusion (PIE) for the case of 3 sets. That is, show that, for any sets A, B, and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

Hint: Observe that $A \cup B \cup C = (A \cup B) \cup C$ and use PIE for two sets, i.e., $|R \cup S| = |R| + |S| - |R \cap S|$

Problem 5. How many integer solutions does the equation $x_1 + x_2 + x_3 = 25$ have, where $0 \leq x_i \leq 10$ for all i?

Problem 6. How many composite numbers are less than 100? Solve this problem without actually listing the numbers!