Homework on Proofs

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Problem 1. A positive integer k (i.e., $k \geq 1$) is said to be a *perfect square* if there is a positive integer i such that $k = i^2$. Prove that for any positive integer n, if n is a perfect square then $n + 2$ is not a perfect square.

Problem 2. Prove that for any real number x, if $\frac{2x+1}{x+1}$ is rational then x is rational.

Problem 3. Let p, q be real numbers such that $q \neq 2$. Prove that if $\frac{2p+1}{q-2}$ is irrational then either p or q is irrational.

Problem 4. Prove that for any real number x, $|x + 3| - x \geq 3$.

Problem 5. An exploration of alternating quantifiers.

a) Prove that, depending on the definition of P, $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ may not have the same truth value. [1] [2]

b) Prove that, regardless of the definition of P, $[\exists x \forall y P(x, y)] \rightarrow [\forall y \exists x P(x, y)]$

Problem 6. Prove that $\sqrt{2} + \sqrt{3} < \sqrt{11}$. [3]

Problem 7. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

[1] A predicate with multiple arguments works just like our one-argument predicates - for example $P(x, y)$ might be defined as "$xy = x + 1$", in which case $P(1, 2)$ is true but $P(4, 4)$ and $P(2, 1)$ are false.

[2] $\exists x \forall y P(x, y)$ is parsed as $\exists x(\forall y(P(x, y)))$

[3] Don't use a calculator! We expect a proof that relies on the algebraic properties of numbers and square roots.