Recursion and Structural Induction Homework

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Problem 1. Consider a set S defined recursively: $1 \in S$, and if $x \in S$ then $-x \in S$ and $2x \in S$.

a) What is the set S? (Give an equivalent non-recursive description, e.g. in set-builder notation or using sets we've named in the past.)

b) A recursive definition for a set is called *ambiguous* if there is any element in the set that can be constructed from the given base cases and constructors in more than one way. [1] Prove the above definition for S is ambiguous.

c) Is it possible to define a recursive function $f : S \to \mathbb{N}$ which counts how many constructor steps were used to produce a given element of S? Why or why not? (For example, we might want $f(1) = 0$ because 1 is in the base case so it looks like it doesn't use any applications of the constructor step.)

d) Come up with a new *unambiguous* recursive definition for the same set S. [2]

[1] Being ambiguous does not make a definition invalid, but it sometimes makes it less useful.

[2] Hint: You may want multiple base case elements instead of just 1.

Problem 2. Consider a set S defined recursively: $-2 \in S$, and if $x \in S$ then $2x \in S$ and $x^2 - 1 \in S$. You may use without proof that every element of S is an integer.

a) Prove $5 \notin S$.

b) Prove $\forall y \in S(|y| \geq 2)$ by structural induction.

c) Prove that for every $y \in S$, if $y < 0$ then y is even.[3]

[3] Hint: use the result from part (b).

Problem 3.

a) Give a recursive definition for a function $rev : \{0,1\}^* \to \{0,1\}^*$ which reverses a binary string. For example, $rev(001111) = 111100$.

b) Prove by structural induction that $rev(st) = rev(t)rev(s)$.

Problem 4. Prove by structural induction: If T is a binary tree, then for any labeling of T's vertices using blue and orange such that the root is blue and every leaf is orange, there exists some blue vertex that has an orange child.