# *Counting*

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TAKE-AWAYS

- Sum rule: If $A_1, \ldots, A_n$ are disjoint, $\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$.
- Product rule: $\left|\bigtimes_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$.
- Bijection rule: If $f : A \to B$ is a bijection, then $|A| = |B|$.
- Generalized product rule: The number of length $k$ sequences where the $i$-th entry has $n_i$ possibilities is $\prod_{i=1}^{k} n_i$.
- Permutation rule: The number of permutations of distinct objects is $n! = n \cdot (n-1) \cdots 2 \cdot 1$. The number of ways to order $k$ objects out of $n$ is $P(n,k) = \frac{n!}{(n-k)!}$.
- Subset/combination rule: The number of ways to choose a $k$-element subset from an $n$-element set is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Discrete structures are often also called "combinatorial" structures, and for good reason: many of their properties are obtained by *combinatorics*, i.e., the art of counting. We have seen examples of this before.

**Example 1.** The Handshaking Lemma for undirected graphs[1] was established by observing that each edge counts towards the degree of two vertices, namely, its two endpoints.

[1] $|E(G)| = \frac{\sum_{v \in V(G)} \deg(v)}{2}$

Most of the "rules" we will cover are actually theorems, but here we will be more interested in applying these concepts and using them in proofs than in actually proving them.

## *Basic Counting*

We will begin with the most basic counting rules.

**Proposition 2** (Sum Rule)**.** *Given n disjoint*[2] *sets* $A_1, \ldots, A_n$,
$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|.$$

[2] i.e., $A_i \cap A_j = \emptyset$ if $i \neq j$

**Example 3.** $|\{1,2,3,4,5,6\}| = |\{1,2,3\} \cup \{4,5,6\}| = |\{1,2,3\}| + |\{4,5,6\}| = 6$.

**Example 4.** Suppose you ask for an explanation on an Examlet problem by posting a private note to Piazza. There are 3 instructors and 7 TAs, for a total of 10 people who could answer your question.

**Example 5.** In a tree, an *internal* vertex is any vertex that is not a leaf. In a binary tree with $i$ internal vertices wherein each internal vertex has exactly two children, the total number of vertices is $2i + 1$. Each vertex is the child of some internal vertex, except for the root, and each internal vertex has exactly two children. So adding 2 to itself $i$ times counts all of the vertices other than the root, which we can account for by adding 1.

**Proposition 6** (Product Rule)**.** *Given n sets* $A_1, \ldots, A_n$,
$\left|\bigtimes_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|$. [3]

[3] $\bigtimes_{i=1}^{n} A_i = A_1 \times \cdots \times A_n$, and $\prod_{i=1}^{n} x_i = x_1 \cdot x_2 \cdots x_n$.

**Example 7.** For a directed graph $G$, $E(G) \subseteq V(G) \times V(G)$. The maximum number of possible edges is thus $|V(G) \times V(G)| = |V(G)|^2$.

**Example 8.** The set of binary sequences of length $k$ is
$\{0,1\}^k = \overbrace{\{0,1\} \times \cdots \times \{0,1\}}^{k \text{ times}}$.
The number of such sequences is $|\{0,1\}|^k = 2^k$.

In addition to using each rule individually, we can also use the Sum and Product rules together.

**Example 9.** Two standard six-sided dice,[4] one orange and one blue, are rolled. How many ways can the sum of the two rolls be even? Well, there are two ways for the sum to be even: both dice rolled odd numbers, or both dice rolled even numbers. Each die has three possible even numbers and three possible odd numbers, so the number of ways to roll two odd numbers is $3 \cdot 3 = 9$, and the number of ways to roll two even numbers is also $3 \cdot 3 = 9$. The two cases cannot happen at the same time, so we add the numbers together to see that the total number of ways to roll an even sum is $9 + 9 = 18$.

[4] Standard here means the sides are labeled 1 through 6.

**Example 10.** Computer password practices used to be quite poor. It was quite common for people to have passwords of length between six and eight where the last character was a number, and the others were all lower-case letters. Since there are 26 lower-case letters and 10 digits, the number of such password of length six is $26^5 \cdot 10$, the number of length seven is $26^6 \cdot 10$, and the number of length eight is $26^7 \cdot 10$. Thus the total number of such passwords is $(26^5 + 26^6 + 26^7) \cdot 10$, which is about $8.35 \times 10^{10}$. This might seem like a lot, but remember that modern computers are quite fast and can brute force this number of possibilities quite quickly![5]

[5] If we take into account the fact that people liked to use words or band names ("blink182" was quite popular), the number of actual possibilities is reduced by quite a bit.

**Proposition 11** (Bijection Rule)**.** *If* $f : A \to B$ *is a bijection,* $|A| = |B|$.

Often the bijection rule is used together with one or more other rules.

**Example 12.** Consider a set $S$ of cardinality $n$, and let $T$ be a subset of $S$. Since $T$ and $S \setminus T$ are disjoint by definition, $|T| + |S \setminus T| = |S|$, so if $|T| = k$, $|S \setminus T| = n - k$. We can define a function $f$ mapping a subset of cardinality $k$ to a subset of cardinality $n - k$ via $f(T) = S \setminus T$. This function is a bijection,[6] so the number of subsets of cardinality $k$ is the same as the number of subsets of cardinality $n - k$.

[6] You should verify this for yourself.

**Example 13.** Consider the function $f : \{0,1\}^k \to \mathcal{P}(\{1,2,\ldots,k\})$ defined by $f((x_1, x_2, \ldots, x_k)) = \{i \mid x_i = 1\}$. $f$ is a bijection,[7] so the number of subsets of $\{1,2,\ldots,k\}$ is $2^k$. More generally, one can always find a bijection between any set $S$ of cardinality $k$ and $\{1,2,\ldots,k\}$, so $|\mathcal{P}(S)| = 2^{|S|}$.

[7] You should also verify this for yourself.

## *Counting Sequences and Subsets*

In a prior example, we considered the case of counting binary sequences, and used the product rule to determine that the number of binary sequences is $2^k$. One way to interpret this result is that there are two possible values for each entry in the sequence. More generally, we can consider the situation where each entry can have some other number of possible values.

**Proposition 14** (Generalized product rule)**.** *The number of length $k$ sequences where the $i$-th entry has $n_i$ possibilities is* $\prod_{i=1}^{k} n_i$.

This is really just the product rule applied to $n_i = |A_i|$, but this alternative viewpoint without the baggage of set notation allows us to think about sequences as more than just elements of Cartesian products. In particular, we have the following example:

**Example 15.** Consider a standard deck of 52 cards. We wish to compute the number of ways to arrange the deck, i.e., the number of possible sequences of cards where no card is repeated. Note that the order matters here. There are 52 possible choices for the first card, but once the first card has been chosen it cannot be repeated. So there are 51 choices for the second card, and so on. In summary, there are $52 \cdot 51 \cdot 50 \cdots 2 \cdot 1 = 52!$ ways of arranging the deck.

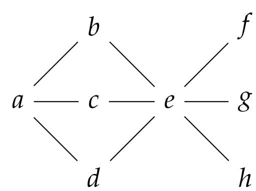
This specific kind of invocation of the Generalized product rule involves a *permutation*: an ordering of $n$ distinct objects in a sequence so that each object appears exactly once. More generally, we may wish to arrange a subset of the $n$ objects as well.

**Example 16.** Suppose 20 people enter a contest. How many ways can we pick a first-place winner, a second-place winner, and a third-place winner? Well, there are 20 choices for the first-place winner, and then 19 choices for second-place, and 18 choices for third-place, for a total of $20 \cdot 19 \cdot 18$.

Note that $20 \cdot 19 \cdot 18 = \frac{20!}{17!}$. More generally, we can summarize the last two examples via the following rule:

**Proposition 17** (Permutation rule)**.** *The number of permutations of distinct objects is $n! = n \cdot (n-1) \cdots 2 \cdot 1$. The number of ways to order $k$ objects out of $n$ is $P(n,k) = \frac{n!}{(n-k)!}$.*

**Example 18.** Consider the following graph $G$. How many isomorphisms are there between $G$ and itself?




Vertices $b$, $c$, and $d$ can be permuted, as can vertices $f$, $g$, and $h$, independently of the permutation of $b, c, d$. $a$ must map to itself, and $e$ must map to itself as well. None of $b, c, d$, can map to $f, g, h$ or vice versa. Combining the permutation rule and the generalized product rule gives $(3!) \cdot (3!) = 36$ possible self-isomorphisms of $G$.

Of course, when working with $n$ distinct objects, we can also simply choose $k$ of them without caring about the order. Let $\binom{n}{k}$ denote the number of ways to choose a $k$-element subset from a set of $n$ elements.[8] We can compute $\binom{n}{k}$ from the permutation rule: once we have chosen the $k$-element subset, there are $k!$ ways of permuting these elements, so $\binom{n}{k}(k!) = P(n,k)$. Dividing both sides by $k!$ gives us the following:

[8] $\binom{n}{k}$ is read as "$n$ choose $k$" and is typeset via `\binom{n}{k}`.

**Proposition 19** (Subset/combination rule)**.** *The number of ways to choose a k-element subset from an n-element set is $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.*

Recall that there is a bijection between the set of $k$-element subsets of an $n$-element set and the set of $(n-k)$-element subsets. This is consistent with the fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$.

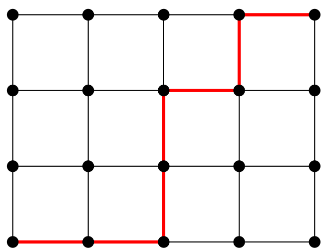
The subset rule is actually quite powerful, and can be applied in some rather interesting ways. We will close out with some more advanced examples of the subset rule being applied.

## *More involved examples*

**Example 20** (Binomial Theorem). $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. When expanding out $(x+y)(x+y)\cdots(x+y)$, we get one instance of the term $x^k y^{n-k}$ from each way of picking $x$ from $k$ of the factors, and $y$ from the remaining $n-k$. There are $\binom{n}{k}$ ways of picking $k$ factors out of $n$.

The special case $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k}$ relates the number of subsets of each cardinality $k$ to the total number of subsets as seen in an earlier example.

**Example 21** (Monotonic paths in a grid). Suppose we are walking on an integer grid. We start at $(0,0)$, and we would like to walk to $(m,n)$, by only ever walking to the right, or upwards. One possible path from $(0,0)$ to $(4,3)$ is shown in the figure below.



If we are only ever allowed to move rightwards or upwards, then we must move right $m$ times and up $n$ times during our path. So each path consists of $m+n$ steps, out of which we pick $m$ to be rightwards (or, equivalently, $n$ to be upwards). In total the number of possible paths is $\binom{m+n}{m} = \binom{m+n}{n}$.

**Example 22** (Integer-sum solutions). Suppose we want to know how many non-negative integer solutions there are to the equation $\sum_{i=1}^{n} x_i = k$, where $k$ is some integer constant. We will model this problem using what is called a "stars-and-bars" formulation. For a non-negative integer solution, we will form a sequence of $k$ stars and $n-1$ bars, where we have $x_1$ stars, then a bar, then $x_2$ stars, then a bar, and so on.

For example, consider the equation $5+2+0+3 = 10$. This can be represented via the following stars-and-bars diagram:

$$\overbrace{\star\ \star\ \star\ \star\ \star}^{5} \mid \overbrace{\star\ \star}^{2} \mid \overbrace{\quad}^{0} \mid \overbrace{\star\ \star\ \star}^{3}$$

The integer values for the variables in a solution can be retrieved by letting $x_i$ count the stars between the $(i-1)$-th bar and the $i$-th bar. This forms a bijection[9] between the set of non-negative integer

[9] As before, you should verify for yourself that this is in fact a bijection!

solutions to $\sum_{i=1}^{n} x_i = k$ and the set of stars-and-bars diagrams with $k$ stars and $n-1$ bars. So counting the number of non-negative integer solutions is the same as counting the number of such stars-and-bars diagrams.

We have a total of $k+n-1$ symbols, out of which we need to pick $n-1$ to be bars (or, equivalently, $k$ to be stars). So the total number of stars-and-bars diagrams with $k$ stars and $n-1$ bars is $\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$.

**Example 23** (Picking from a set with repeats)**.** Suppose we have a set of $n$ elements, and we want to pick $k$ elements, potentially with repeats. For example, a fruit seller might sell four kinds of fruit: apples, bananas, cantaloupes, and durians, and we want to buy a total of ten fruits. One possibility is to buy five apples, two bananas, zero cantaloupes, and three durians. How many ways can we do so?

We can always find a bijection between a set of $n$ elements and the set $\{1, 2, \ldots, n\}$, and then view this problem as finding a non-negative integer solution to the equation $\sum_{i=1}^{n} x_i = k$, where $x_i$ represents the number of times we take element $i$. But we already solved this problem in the previous example: the answer is $\binom{k+n-1}{n-1}$ $(= \binom{k+n-1}{k})$.