Directed Graphs

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TAKE-AWAYS

- A *directed graph* (or *digraph*) is a set of *vertices* along with a set of directed *edges* that each point from one vertex to another.
- Informally, a digraph is just a bunch of dots and arrows, which can be a useful visualization of a relation $R \subseteq A \times A$.
- A *walk* is an alternating list of vertices and the edges that connect them (usually described using just the vertices or just the edges). A *path* is a walk that does not repeat vertices. A *closed walk* is a walk that ends where it begins. A *cycle* is a positive-length closed walk that does not repeat vertices (other than the end being a repeat of the start).
- $dist(u, v)$ (short for *distance*) is the length of the shortest path from u to v.
- Two walks can be *merged* into one walk $w_1\widehat{x}w_2$ if w_1 ends at x and w_2 starts at x.
- A *DAG* (directed *acyclic* graph) is a digraph with no cycles. A *cyclic* digraph has one or more cycles.
- A *topological sort* of a digraph is a list of all its vertices such that for each edge (u, v), u comes before v in the list. On this week's homework you will prove that a digraph has a topological sort if and only if it is a DAG.
- A relation R is *reflexive* if $\forall x(xRx)$.
- A relation R is *symmetric* if $\forall x \forall y(xRy \to yRx)$.
- A relation R is *transitive* if $\forall x \forall y \forall z((xRy \land yRz) \to xRz)$.
- A relation is an *equivalence relation* if it is reflexive, symmetric, and transitive.

Directed Graphs

WHAT does the internet have in common with your course catalog? Both can be thought of as a group of things with connections between them. On the internet, we have web pages, and some are connected to each other by hyperlinks; in the catalog we have classes, and some classes are required by others as prerequisites. And some of the questions we might ask about both internet and catalog are based entirely on this structure. On the internet, we might want to identify pages with lots of incoming links because they're more likely to be trustworthy. In the catalog, you might want to identify classes with many outgoing links (i.e. which are often required as prerequisites) so you can take them early and have more choices later. So let's do the mathematician thing and zoom in on the underlying structure for closer study.

Definition 1 (Directed Graphs)**.** A *directed graph* (*digraph* for short) is a set of *vertices* $V(G)$[1] and a set of *directed edges* $E(G)$, where $E(G) \subseteq V(G) \times V(G)$.

[1] In this class you may always assume a graph has a *finite* number of vertices unless a problem specifies otherwise

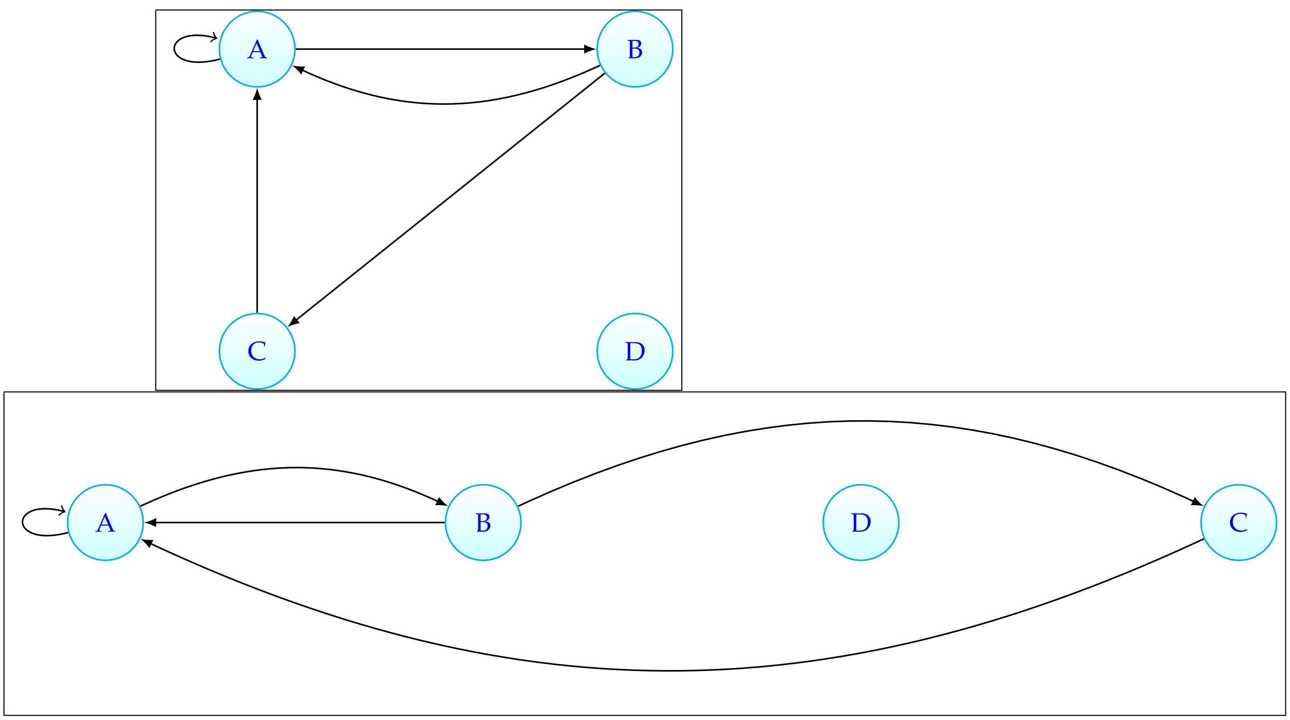

Figure 1: Two representations of the exact same digraph G where $V(G) = \{A, B, C, D\}$ and $E(G) = \{(A, A), (A, B), (B, A), (B, C), (C, A)\}$

We will often represent a digraph using a picture by drawing a dot for each vertex and an arrow pointing from u to v for each edge

(u, v) (Figure 1). The picture is a valid representation of the digraph as long as the correct vertices and arrows are present; other details of the drawing, such as the relative positions of the vertices or curvature of the arrows, do not matter.

Many questions involving digraphs end up being about "walking around" on the graph from vertex to vertex across edges - sometimes literally, if the vertices are locations and the edges are (one-way) roads, but also metaphorically, such as following a chain of prerequisites through several classes. We formalize this idea as follows:

Definition 2 (Walks)**.** A *walk* in a digraph is an alternating sequence of vertices and edges starting and ending with a vertex, such that each edge points from the vertex before it to the one after it in the sequence. The *length* of a walk w, written $|w|$, is the number of edges in it.

Example 3. In Figure 1, the following is a walk of length 4:
$C(C, A)A(A, A)A(A, A)A(A, B)B$

You will rarely see (or have to write) a walk in its official full form - as long as it is clear from context, it is common to abbreviate it to just the list of vertices, or occasionally just the list of edges. So in the example above, we would more frequently write $CAAAB$[2] or perhaps $(C, A)(A, A)(A, A)(A, B)$.

[2] Note that the *length* of the walk is still the number of *edges*, so the length of $CAAAB$ is 4, not 5

Definition 4 (Merging walks)**.** Two walks w_1 and w_2 can be *merged* into one walk if the second starts at the same vertex x where the first ended. We write the merged walk as $w_1\widehat{\ }w_2$ or $w_1\widehat{x}w_2$.

Example 5. In Figure 1, if w_1 is AB and w_2 is BAC then $w_1\widehat{\ }w_2$ (or $w_1\widehat{B}w_2$) is $ABAC$.

In several application domains, we frequently want to know if a walk repeats any vertices and if it ends up back where it started. For example for a physical walk out in the city, visiting the same location multiple times may be undesirable if you're trying for some kind of variety or errand efficiency, and if you start at home you may want to make sure you end up there again at end of day. We formalize these ideas as follows (sorry for the flood of vocabulary this week!):

Definition 6 (Walk variants)**.** A *path* is a walk that does not repeat vertices. A *closed walk* is a walk that ends where it starts. A *cycle* is a positive-length closed walk that does not repeat vertices (except the end vertex, which does need to be a repeat of the start).

Example 7. In Figure 1, A and CAB are paths. $CABCABC$ is a closed walk, but not a cycle because of repeat vertices. A is a closed walk but not a cycle because it has length 0. AA and $ABCA$ are closed walks and also cycles.

It is sometimes easier to work with walks in G using a few related graphs. For each $n \in \mathbb{N}$, let G^n be the digraph where $V(G^n) = V(G)$ and $(u, v) \in E(G^n)$ when there is a walk of length exactly n from u to v in G. Similarly, an edge (u, v) is in $E(G^*)$ if there is *any* walk from u to v in G, and (u, v) is in $E(G^+)$ if there is any *positive-length* walk from u to v in G. See Figure 2 for examples.

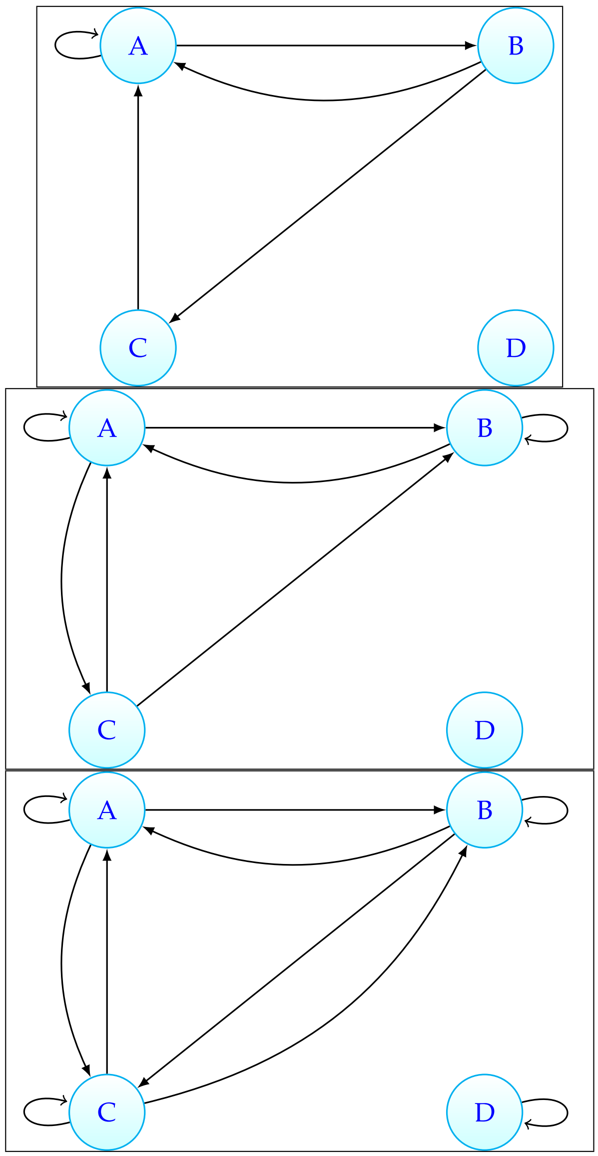

Figure 2: From top to bottom: a) The original graph G from Figure 1, b) G^2, and c) G^*. G^+ would be the same as G^* except that it would not have the self-loop on vertex D (because in G there is a walk from D to D of length 0, but no walk with positive length). Notice that not all edges from G appear in G^2.

Definition 8 (Distance)**.** The *distance* from vertex u to v, written $dist(u, v)$, is the length of any *shortest path* from u to v.

Example 9. In Figure 1, $dist(B, B) = 0$, $dist(B, C) = 1$, and $dist(C, B) = 2$.

In many domains it is important that a digraph have no cycles at all. For example, if there were any cycles in the course prerequisite graph, it would be impossible to take any of those courses!

Definition 10 (DAG)**.** A digraph that has no cycles is *acyclic*; a digraph with any cycles is called *cyclic*. "Acyclic digraph" is often abbreviated *DAG* (Directed Acyclic Graph).

A key property related to DAGs is the topological sort:

Definition 11 (Topological sort)**.** A *topological sort* of a digraph G is an ordering of all the vertices $V(G)$ such that for each edge (u, v), u appears before v in the ordering.

Example 12. Figure 3 is a DAG because there are no cycles. The two possible topological sorts are A, B, C, D and A, B, D, C.

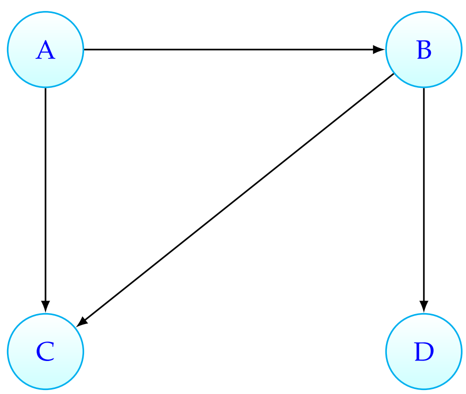

Figure 3: An example DAG (directed acyclic graph)

The following important link between DAGs and topological sorts is listed here for completeness but do not use this result on the worksheet or homework since you will be proving it yourself:

Theorem 13. *A directed graph has a topological sort if and only if it is acyclic.*

Relations

Notice that for any digraph G, the set of edges $E(G)$ is a *relation*[3]. Similarly, any binary relation $R \subseteq A \times A$ can be turned into the edges of a digraph. (Relations on two *different* sets, or on three or more sets, are not naturally turned into graphs.)

An common and important kind of relation is one that represents when two things are in some way "the same". Examples include

[3] Review week 3 if needed

equality (e.g. $3 = 2 + 1$) and logical equivalence between propositions (e.g. $\neg\neg p \equiv p$). There are three properties which together correspond to an intuitive notion of how a "sameness" relation should behave: every item is the same as itself, if x is the same as y then y is the same as x, and if x is the same as y and y is the same as z then x is the same as z. We formalize these properties as follows:

Definition 14 (Equivalence Relations)**.** A relation $R \subseteq A \times A$ is

- *reflexive* if $\forall x(xRx)$[4].
- *symmetric* if $\forall x \forall y(xRy \rightarrow yRx)$.
- *transitive* if $\forall x \forall y \forall z((xRy \wedge yRz) \rightarrow xRz)$.
- an *equivalence relation* if it is reflexive, symmetric, and transitive.

[4] Recall: we say aRb to mean $(a, b) \in R$

Example 15. Consider the relation $R \subseteq \mathbb{R} \times \mathbb{R} = \{(a, b) | a \leq b\}$. It is reflexive because for any x, $x \leq x$. It is transitive because for any x, y, z where $x \leq y$ and $y \leq z$, we also have $x \leq z$. However it is not symmetric: even though there are some x, y where $(xRy \rightarrow yRx)$ holds (i.e., any x and y that are equal), for R to be symmetric that would need to hold for *every* x, y and it does not (e.g. $3 \leq 5$ but $5 \not\leq 3$). Thus R is not an equivalence relation (which makes sense according to the "sameness" intuition - $x \leq y$ was never supposed to mean that x and y must be in any way the same).