Sets, Functions, and Relations

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TAKE-AWAYS

- A set is an unordered collection of objects, typically listed without repetition.
- $\emptyset$ is the emptyset, $\mathbb{Z}$ the set of integers, $\mathbb{N}$ the set of natural numbers (or non-negative integers), $\mathbb{R}$ the set of real numbers, and $\mathbb{Q}$ the set of rational numbers.
- Elements of a set uniquely determine the set. $x \in S$ denotes that x is an element of set S.
- $R \subseteq S$ denotes that R is a subset of S or that every element in R is also an element of S. $R = S$ if $R \subseteq S$ and $S \subseteq R$.
- For sets R and S, $R \cup S$ denotes their union, $R \cap S$ denotes their intersection, $R \setminus S$ denotes their difference, and $\overline{R}$ denotes the complement of R (with respect to a universe).
- The Cartesian product of sets R and S is the set of all ordered pairs of the form (r, s) where $r \in R$ and $s \in S$.
- The power set of a set S is the set consisting all subsets of S.
- For a function $f : A \to B$, A is the domain, and B is the codomain of function f. The range of function f is the set $\{y \in B \mid \exists x \in A \; f(x) = y\}$.
- A function f is surjective/onto if $\mathsf{codom}(f) = \mathsf{rng}(f)$.
- A function f is injective/1-to-1 if distinct elements are mapped to distinct elements.
- A function f is bijective if it is both injective and surjective.
- Binary relations, with domain A and co-domain B, are subsets of $A \times B$.

Sets

A SET is an *unordered* collection of objects, typically listed without repetition between braces. Unordered means that the way the elements of a set are listed does not matter. For example, the set $\{0, 2, 4, 6\}$ is the same as the set $\{2, 0, 6, 4\}$. Elements of a set are also listed only once. So the set $\{0, 0\}$, is the same as the set $\{0\}$.

Notation. There are some important sets and their notation that you should be comfortable with.

- $\{\}$ and $\emptyset$ denote the *empty set* which is the unique set that contains no elements.
- $\mathbb{Z} = \{\ldots - 2, -1, 0, 1, 2, \ldots\}$ denotes the set of all *integers*.
- $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of *natural numbers*. Notice that in this course, and typically in computer science/mathematics, this set includes the number 0. If you are confused by this, you may want to read and remember $\mathbb{N}$ as $\mathbb{N}$*on-negative integers*.
- $\mathbb{R}$ denotes the set of all real numbers.

Example 1. Sets can contain any elements as members, including numbers, letters, symbols, or even other sets. Here are some examples.

$$\begin{array}{ll} A = \{0, 2, 4, 6\} & P = \{\mathsf{B}, \mathsf{C}, \mathsf{D}, \mathsf{E}, \mathsf{F}, \mathsf{J}, \mathsf{K}, \mathsf{P}, \mathsf{Q}, \mathsf{R}, \mathsf{S}, \mathsf{T}, \mathsf{V}\} \\ B = \{\{0\}, \{2\}, \{4\}, \{6\}\} & C = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} \end{array}$$

The set P above is a set of names of programming languages. How many of them did you know?

Since the elements in a set are unordered and do not repeat, a set is completely determined by its members. The "$x \in S$" means that x *is a member of set* S. An element of a set is something that is written between commas after erasing the outermost braces. Let us look at some examples.

Example 2. Let us look at the members of some sets introduced in Example 1. The elements of set $A = \{0, 2, 4, 6\}$ are $0, 2, 4$, and 6. On the other hand, the elements of set $B = \{\{0\}, \{2\}, \{4\}, \{6\}\}$ are $\{0\}$, $\{2\}$, $\{4\}$, and $\{6\}$. Notice, that $0 \notin B$ but $\{0\} \in B$. Similarly, $0 \in A$ but $\{0\} \notin A$.

The members of set $C = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ are $\emptyset$, $\{\emptyset\}$, and $\{\emptyset, \{\emptyset\}\}$. Observe that $\emptyset$ is not a member of A or B. But $\emptyset \in C$.

Definition 3 (Containment and Equality)**.** A set A is a *subset of / contained in* B, denoted $A \subseteq B$, if every element of A is also an element of B. That is, $\forall x\ (x \in A \to x \in B)$.

Two sets A and B are *equal*, denoted $A = B$, if every element of A is an element of B, and vice versa. That is, $A \subseteq B$ and $B \subseteq A$. Another way to say this is that $\forall x\ (x \in A \leftrightarrow x \in B)$.

Let us look at some examples.

Example 4. Let us consider the sets $\emptyset$, $\mathbb{N}$, $A = \{0, 2, 4, 6\}$, $B = \{\{0\}, \{2\}, \{4\}, \{6\}\}$.

Observe that $\emptyset \subseteq A$. Let us see why this is the case. We need to prove that any element belonging to $\emptyset$ also belongs to A. However, there are no elements in $\emptyset$, and so this property holds vaccuously. Similar reasoning allows one to show that $\emptyset \subseteq B$ and $\emptyset \subseteq \mathbb{N}$.

Next, we can also show that $\mathbb{N} \subseteq \mathbb{N}$. This observation also follows from the definition of set containment — every element that belongs to $\mathbb{N}$ also (by definition) belongs to $\mathbb{N}$; hence, $\mathbb{N} \subseteq \mathbb{N}$. We can similarly, show that $A \subseteq A$, $B \subseteq B$, and $\emptyset \subseteq \emptyset$.

Finally, observe that $A \not\subseteq B$. This can be seen by observing that $0 \in A$, but $0 \notin B$. Similarly, $B \not\subseteq A$ as $\{0\} \in B$ but $\{0\} \notin A$.

The observations made in Example 4 can be encapsulated in a couple of simple propositions.

Proposition 5. *For any set S, $\emptyset \subseteq S$.*

Proof. Let S be an arbitrary set. To show that $\emptyset \subseteq S$, we need to prove $\forall x\ (x \in \emptyset \to x \in S)$. But since "$x \in \emptyset$" is not true for any element x, the implication $(x \in \emptyset \to x \in S)$ holds vaccuously, for any x. Thus, we have proved the implication, and therefore, $\emptyset \subseteq S$. $\square$

Proposition 6. *For any set S, $S \subseteq S$.*

Proof. Let S be an arbitrary set. To show that $S \subseteq S$, we need to prove $\forall x\ (x \in S \to x \in S)$. We will prove this using a direct proof. Let x be an arbitrary element such that $x \in S$. This means (trivially) that $x \in S$, and hence the implication holds. Therefore, $S \subseteq S$. $\square$

One of the most important ways to define a subset of a given universal set, is using the set builder notation. Many sets are defined in this manner.

Definition 7 (Set Builder)**.** The notation $\{x \in S \mid P(x)\}$ defines the subset of S consisting of elements that satisfy the predicate P.

Let us look at some examples.

Example 8. The set of even integers can be defined as follows.

$$E = \{n \in \mathbb{Z} \mid n \text{ is even}\} = \{n \in \mathbb{Z} \mid \exists m \in \mathbb{Z}\ n = 2m\}$$

The set $\mathbb{Q} = \{r \in \mathbb{R} \mid \exists m, n \in \mathbb{Z}\ (n \neq 0 \wedge r = \frac{m}{n})\}$ defines the set of real numbers that can be written as the ratio of two integers, where the denominator is non-zero. In other words, $\mathbb{Q}$ is the set of all rational numbers.

Set Operations

SETS CAN COMBINED in different ways to create new sets. We will describe some standard set operations in this section. We begin with the Boolean operations of union, intersection, difference, and complement.

Definition 9. Let R and S be arbitrary sets. Union ($R \cup S$), intersection ($R \cap S$), difference ($R \setminus D$), and complementation are defined as follows.

$$R \cup S = \{x \mid x \in R \text{ or } x \in S\} \quad R \cap S = \{x \mid x \in R \text{ and } x \in S\}$$
$$R \setminus S = \{x \in R \mid x \notin S\}$$

Typically there is a universe/domain of discourse that all sets in a discussion are subsets of. This universe is often implicitly known in the context. In such a case, the *complement* of set R (with respect to universe U) is given as $\overline{R} = U \setminus R$.

Boolean operations on sets are best understood through what is often called a *Venn diagram*, that shows logical relationships between different sets. Sets are often represented as circles with their spatial arrangement mimicing their relationships. The universe is typically shown as a rectangle enclosing all the sets. Union and intersection of sets R and S can be pictorial shown in a Venn diagram as the shaded regions in Figure 1 and Figure 2, respectively.

Figure 1: $R \cup S$

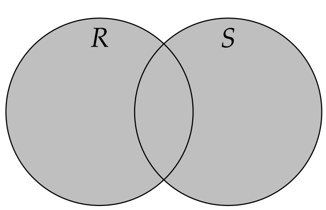
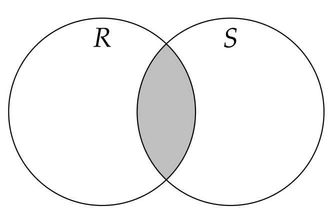


Figure 2: $R \cap S$



Similarly, the difference between R and S, and their complement are shown in Figure 3 and Figure 4.

Let us look at some examples.

Figure 3: $R \setminus S$

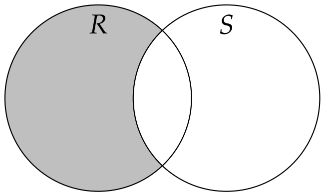
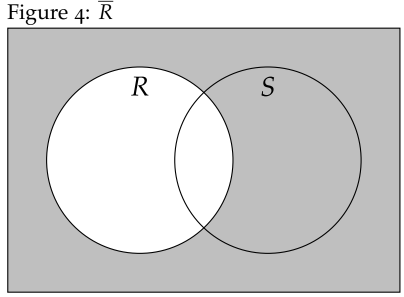


Figure 4: $\overline{R}$



Example 10. Let us recall the sets $A = \{0,2,4,6\}$, $B = \{\{0\},\{2\},\{4\},\{6\}\}$, from Example 1. Observe that $A \cup B = \{0,2,4,6,\{0\},\{2\},\{4\},\{6\}\}$ but $A \cap B = \emptyset$. Further, $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, and $A \setminus B = A$. On the other hand, $B \setminus A = B$. Thus, in general, for sets R and S, $R \setminus S$ is not necessarily equal to $S \setminus R$.

Example 10 makes a couple of observations that hold for any sets (not just A and B of the example).

Proposition 11. *For any set S, $S \cup \emptyset = S$ and $S \cap \emptyset = \emptyset$.*

The set operations of union, intersection, and complementation, satisfy many of the properties that $\vee$, $\wedge$, and $\neg$ satisfy in logic. The proof that they satisfy these properties often exploits the same properties of the logical operators. Let us look at one example to illustrate this idea.

Proposition 12. *For any sets X, Y, Z, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.*

Proof. Let us fix some arbitrary sets X, Y, and Z.

Recall that proving the equality of two sets R and S involves arguing that element of the first set is also an element of the second set and vice versa. As in the proof of any if and only if statement, there are two implications to be proved, and we need to do that in set equality proofs. In this case each direction can be proved by a direct proof.

$X \cap (Y \cup Z) \subseteq (X \cap Y) \cup (X \cap Z)$: We need to prove that for any x, if $x \in X \cap (Y \cup Z)$ then $x \in (X \cap Y) \cup (X \cap Z)$. We will use a direct proof to establish this implication.

Let x be an arbitrary element and assume that $x \in X \cap (Y \cup Z)$. We use the definition of the set operations and logical reasoning to observe the following sequence of steps.

$$\begin{aligned}
x \in X \cap (Y \cup Z) \quad &\to \quad (x \in X) \wedge (x \in Y \cup Z) \\
&\to \quad (x \in X) \wedge ((x \in Y) \vee (x \in Z)) \\
&\to \quad ((x \in X) \wedge (x \in Y)) \vee ((x \in X) \wedge (x \in Z)) \\
&\to \quad (x \in X \cap Y) \vee (x \in X \cap Z) \\
&\to \quad (x \in (X \cap Y) \cup (X \cap Z))
\end{aligned}$$

$(X \cap Y) \cup (X \cap Z) \subseteq X \cap (Y \cup Z)$*:* This can also be proved by a direct proof. The proof is just the reverse of the sequence of steps used in the argument for the other direction. We skip repeating the steps, and leave it to the reader as an exercise.

In proofs like the one above where the proof in each direction is the same set of steps but in reverse order, it is often written as follows to emphasize that each step in the argument can be reversed to get a logical sound sequence of steps.

$$\begin{aligned} x \in X \cap (Y \cup Z) &\leftrightarrow (x \in X) \wedge (x \in Y \cap Z) \\ &\leftrightarrow (x \in X) \wedge ((x \in Y) \vee (x \in Z)) \\ &\leftrightarrow ((x \in X) \wedge (x \in Y)) \vee ((x \in X) \wedge (x \in Z)) \\ &\leftrightarrow (x \in X \cap Y) \vee (x \in X \cap Z) \\ &\leftrightarrow (x \in (X \cap Y) \cup (X \cap Z)) \end{aligned}$$

□

The next important set operation is that of Cartesian products.

Definition 13 (Cartesian Products)**.** The *Cartesian product* of two sets R and S (denoted $R \times S$) consists of the set of all ordered pairs (r, s) where $r \in R$ and $s \in S$. Using the set builder notation, this is written as shown.

$$R \times S = \{(r, s) \mid r \in R \text{ and } s \in S\}$$

More generally, for sets $A_1, A_2, \ldots A_n$,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_, \text{ for all } i\}.$$

Let us look at some examples.

Example 14. Consider $R = \{1, 2, 3\}$ and $S = \{a, b, c\}$. Then $R \times S = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$.

Let us look at another example. Consider $U = \{1\}$ and $V = \{a\}$. Then $U \times V = \{(1, a)\}$. On the other hand, $V \times U = \{(a, 1)\}$. Therefore, $U \times V \neq V \times U$. In general, Cartesian product is not a *commutative* operation.

Definition 15 (Power Set)**.** The *power set* of a set S, denoted $\mathcal{P}(S)$, is the set consisting of every subset of S. In other words,

$$\mathcal{P}(S) = \{A \mid A \subseteq S\}.$$

Example 16. $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Note that from Proposition 5, we can conclude that for any set S, $\emptyset \in \mathcal{P}(S)$ and $S \in \mathcal{P}(S)$.

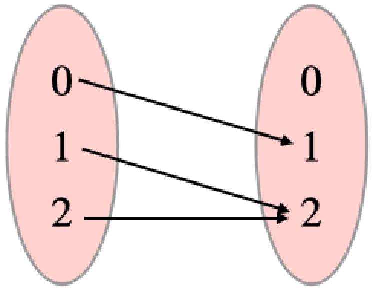


Figure 5: Pictorial representation of the function $g(0) = 1$, $g(1) = 2$, and $g(2) = 2$.

Functions

A FUNCTION $f : A \to B$ assigns an element of B to each element of A. A is the *domain*, and B is the *codomain* of the function f. Given a function f, we will use $\mathsf{dom}(f)$ to denote the domain, and $\mathsf{codom}(f)$ to denote the codomain. On the other hand, the *range* of the function f (denoted $\mathsf{rng}(f)$) is the set

$$\mathsf{rng}(f) = \{b \in B \mid \exists a \in A\ f(a) = b\}.$$

Example 17. Functions can be described in multiple ways. They can be given by explicitly listing the mapping. For example, consider the function $g : \{0,1,2\} \to \{0,1,2\}$ given by

$$g(0) = 1 \quad g(1) = 2 \quad g(2) = 2$$

Sometimes it is convenient to represent the function pictorially, where arrows from the domain to the co-domain describe the mapping. For example, the function g described above is shown pictorially in Figure 5. Observe that $\mathsf{dom}(g) = \mathsf{codom}(g) = \{0,1,2\}$. On the other hand, $\mathsf{rng}(g) = \{1,2\}$; pictorially, these are all the elements that have an incoming arrow.

Most of the time it is convenient to describe the function mathematically. For example, consider the function $\mathsf{dbl} : \mathbb{N} \to \mathbb{N}$ given by $\mathsf{dbl}(n) = 2n$. The domain, codomain, and range of dbl are as follows: $\mathsf{dom}(\mathsf{dbl}) = \mathsf{codom}(\mathsf{dbl}) = \mathbb{N}$, and $\mathsf{rng}(\mathsf{dbl}) = \{2n \mid n \in \mathbb{N}\}$.

Definition 18 (Injective, Surjective, and Bijective Functions)**.** Consider a function $f : A \to B$.

f is said to be *surjective* or *onto* if $\mathsf{rng}(f) = \mathsf{codom}(f)$. That is

$$\forall y \in B \exists x \in A\ f(x) = y.$$

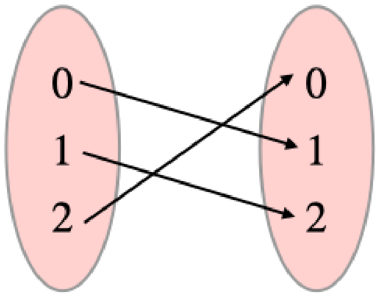


Figure 6: Function f given by: $f(0) = 1$, $f(1) = 2$ and $f(2) = 0$.

f is said to be *1-to-1* or *injective* if distinct elements in A get mapped to distinct elements in B. That is

$$\forall x \in A \forall y \in A\ (x \neq y \rightarrow f(x) \neq f(y)).$$

f is said to be *1-to-1 and onto* or *bijective* if it is both 1-to-1 and onto.

Let us look at some examples.

Example 19. The requirements of onto and 1-to-1, can be understood pictorially as follows. Every element of the codomain has at least one incoming arrow when the function is onto. On the other hand, every element of the codomain as at most one incoming arrow, when the function is 1-to-1. Consider functions $f : \{0,1,2\} \rightarrow \{0,1,2\}$ and $g : \{0,1,2\} \rightarrow \{0,1,2\}$ defined as

$$\begin{array}{lll} f(0) = 1 & f(1) = 2 & f(2) = 0 \\ g(0) = 1 & g(1) = 2 & g(2) = 2 \end{array}$$

Function g is shown in Figure 5, while function f is shown in Figure 6. The function f is both 1-to-1 and onto. On the other hand g is neither 1-to-1 nor onto — g is not onto because $0 \notin \mathsf{rng}(g)$ and it is not 1-to-1 because $g(1) = g(2)$. Since f is both 1-to-1 and onto, it is bijective. On the other hand, g is not bijective.

When proving whether a function is onto or 1-to-1, we use the formal definition of these properties. Notice that injectiveness or 1-to-1ness requires one to prove an implication. Often the most convenient way to establish this is by looking at its contrapositive. Let us look at a couple of examples that state some useful properties about functions. Before looking at these properties, let us define what it means to compose two functions.

Definition 20. For functions $f : A \rightarrow B$ and $g : B \rightarrow C$, their composition $g \circ f$ is a function $A \rightarrow C$ defined as

$$g \circ f(a) = g(f(a)).$$

Proposition 21. *For functions $f : A \to B$ and $g : B \to C$ if f and g are onto/surjective then $g \circ f$ is onto/surjective.*

Proof. Let f and g be arbitrary surjective functions as in the statement. We need to prove that for every $y \in C$ there is some $x \in A$, such that $g \circ f(x) = y$.

Let y be an arbitrary element of C. Since g is surjective, there is $z \in B$ such that $g(z) = y$. Similarly, since f is surjective, there is $x \in A$ such that $f(x) = z$. Observe that $g \circ f(x) = g(f(x)) = g(z) = y$. Thus, $g \circ f$ is surjective. □

Proposition 22. *For functions $f : A \to B$ and $g : B \to C$ if f and g are 1-to-1/injective then $g \circ f$ is 1-to-1/injective.*

Proof. Let f and g be arbitrary injective functions as in the statement. We need to prove that for any $x, y \in A$, if $x \neq y$ then $g \circ f(x) \neq g \circ f(y)$. The contrapositive of this statement is

$$\forall x, y \in A\ (g \circ f(x) = g \circ f(y) \to x = y).$$

We will prove this contrapositive statement. Most proofs showing functions to be injective rely on proving the contrapositive of the definition.

As in any direct proof by contraposition, assume x, y are arbitrary elements of A such that $g \circ f(x) = g \circ f(y)$. Observe that

$$g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y).$$

Since g is injective and $g(f(x)) = g(f(y))$, the arguments to g, namely $f(x)$ and $f(y)$, must be equal. That is, $f(x) = f(y)$. Similarly, since f is injective, we can conclude $x = y$. This proves that $g \circ f$ is injective. □

Relations

A k-ary relation R is a set of k-tuples, i.e., $R \subseteq A_1 \times A_2 \times \cdots \times A_k$. Most of the time, we care about binary relations, which are of the form $R \subseteq A \times B$. For such binary relations, A is called the *domain*, and B is called the *codomain*.

Notation. The following are equivalent ways to say that a and b are related by binary relation R: $(a, b) \in R$, aRb, and $R(a, b)$.

Binary relations are a generalization of functions. Every function $f : A \to B$ has an binary relation associated with it, called the *graph of function f*, denoted $\mathsf{graph}(f)$, that uniquely defines it. Formally, $\mathsf{graph}(f) \subseteq A \times B$ such that $\mathsf{graph}(f) = \{(x, f(x)) \mid x \in A\}$. For example, for the function g (Example 17 and Figure 5), $\mathsf{graph}(g) = \{(0,1), (1,2), (2,2)\}$.