Worksheet on Cardinality

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Definitions from the Lecture

- The cardinality of a finite set A (denoted $|A|$) is the number of elements in set A.
- The cardinality of the Cartesian product of finite sets is the product of the cardinalities of the individual sets, i.e., $|A_1 \times A_2 \times \cdots \times A_k| = n_1 n_2 \cdots n_k$, where $|A_i| = n_i$ for $i \in \{1, 2, \ldots k\}$.
- For finite sets A, B, if there is a surjective function $f : A \to B$ then $|B| \leq |A|$, and if there is a bijective function $f : A \to B$ then $|A| = |B|$.
- For any finite set A, $|\mathcal{P}(A)| = 2^{|A|}$.
- *Cantor's Definition:* For infinite sets A, B, we say $|B| \leq |A|$ if there is a surjective (onto) function $f : A \to B$, and we say $|A| = |B|$ if there is a bijective function $f : A \to B$.
- The following properties hold for Cantor's definition. For any set A, $|A| = |A|$. If $B \subseteq A$ then $|B| \leq |A|$. Finally, for infinite sets A, B, C, if $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$, and if $|A| \leq |B|$ and $|B| \leq |C|$ then $|A| \leq |C|$.
- *Cantor-Schröder-Bernstein Theorem:* For any infinite sets A and B, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.
- For infinite sets A and B, if there is an injective function $f : A \to B$ then there is a surjective function $g : B \to A$. Thus, if there is an injective function $f : A \to B$ then $|A| \leq |B|$.
- A set S is *countable* if either S is finite or $|S| = |\mathbb{N}|$.
- The sets $\mathbb{E}$ $(= \{2n \mid n \in \mathbb{N}\})$, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{N} \times \mathbb{N}$ are all countable.
- $\mathcal{P}(\mathbb{N})$ is not countable.

Problem 1. For each of the following pairs of sets A, B, determine if there is a function $f : A \to B$ that is surjective but not bijective and if there is a function $g : A \to B$ that is bijective.

(a) $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $B = \{0, 2, 4, 6, 8\}$.

(b) $A = \mathbb{N}$ and $B = \mathbb{E} = \{2n \mid n \in \mathbb{N}\}$.

Problem 2. Consider the function $\mathsf{sgn} : \mathbb{Z} \to \mathbb{N}$ that maps non-negative numbers to the even natural numbers and the negative numbers to the odd numbers as follows.

$$\begin{aligned} 0 &\mapsto 0 \\ -1 &\mapsto 1 \\ 1 &\mapsto 2 \\ -2 &\mapsto 3 \\ 2 &\mapsto 4 \\ &\vdots \end{aligned}$$

(a) Give a precise mathematical definition of sgn.

(b) Prove that sgn is bijective.

Problem 3. Prove that for any sets A and B with $A \neq \emptyset$, if there is an injective function $f : A \to B$ then there is a surjective function $g : B \to A$.

Problem 4. Recall that $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$ is the set of rational numbers. Prove that $|\mathbb{Q}| = |\mathbb{N}|$, i.e., $\mathbb{Q}$ is countable. *Hint:* Use the Cantor-Schröder-Bernstein theorem and Problem 3.

Problem 5. Cantor's diagonalization argument (see lecture notes) can be used to prove that $|\mathbb{N}| \neq |\mathcal{P}(\mathbb{N})|$. Use the same proof template to prove that for *any* infinite set A, $|A| \neq |\mathcal{P}(A)|$.