Worksheet on Simple Graphs and Trees

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TAKE-AWAYS

- A *simple graph* is a set of vertices along with a set of (undirected) edges with no self-loops.
- Most concepts from directed graphs, like walks and paths, can be used almost unchanged for simple graphs. A cycle in a simple graph must be of length at least 3.
- The *degree* of a vertex is the number of edges *incident* to it. Two vertices connected by an edge are *adjacent*.
- A graph is *connected* if there is a path from each vertex to each other vertex.
- A connected acyclic graph is called a *tree*; its degree-1 vertices are *leaves*.
- A tree can also be directed; edges point from *parent* to *child* away from the *root*, and the *height* is the maximum distance from the root to a leaf.
- Two graphs are *isomorphic* if they have the same number of vertices, all connected in the same way.
- Some common graphs are the n-vertex line graph L_n, the n-vertex cycle graph C_n, the $(n+1)$-vertex wheel graph W_n, and the n-vertex complete graph K_n.
- A *k-coloring* in a graph is an assignment of k colors to vertices so that adjacent vertices always have different colors.
- A graph's *chromatic number* χ is the smallest number of colors needed to color it.
- A *bipartite* graph is one whose vertices can be split into two non-empty groups and all edges go from one group to the other.
- A *matching* in a bipartite graph is a way of pairing up all vertices in one group with adjacent vertices in the other.
- Colorings and matchings are useful for solving different kinds of allocation problems.

Problem 1. What is the average degree of the vertices in a graph G? (answer in terms of $|V(G)|$ and $|E(G)|$)[1]

[1] Hint: First compute the *sum* of the vertex degrees - how does each edge contribute to that sum? Hint 2: Handshaking Lemma (MCS Lemma 12.2.1)

Problem 2. Find the chromatic number for each graph and briefly justify your answers:

a) K_n

b) $L_n (n \geq 2)$

c) $C_n (n \geq 3)$

d) $W_n (n \geq 3)$

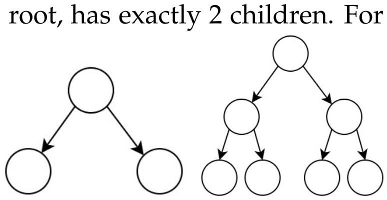
Problem 3. Prove that a graph G with at least two vertices is bipartite iff $\chi(G) \leq 2$.

Problem 4. Let the *signature* of a graph be the list of all vertex degrees in descending order. For example, the signature of L_4 is $2, 2, 1, 1$.

a) Briefly argue that any two isomorphic graphs have the same signature.

b) Prove that the converse is false, i.e. prove that there are some graphs that have the same signature but are not isomorphic.

Problem 5. For $n \geq 1$, define T_n as the "full binary tree" of height n, where each vertex, other than the leaves which are all distance n from the root, has exactly 2 children. For example, here are T_1 and T_2:



a) Fill in the blank, and justify your answer: For any $k > 1$, if x is the number of leaves in T_{k-1} then T_k has ______ leaves.

b) How many leaves does T_k have? (Briefly justify your answer.)[2]

[2] Hint: Start with the number of leaves in T_1 and apply the lemma from (a) as many times as needed.

c) Let k be some fixed natural number greater than 1, and assume that T_{k-1} has $2^k - 1$ vertices. How many vertices does T_k have?[3]

[3] Hint: Use part b. In the most straightforward solution to this part and the next, you don't have to care at all if the assumption is true or false.

d) Let k be some fixed natural number greater than 1, and assume that T_{k-1} has $2^k + 1$ vertices. How many vertices does T_k have?

e) How many vertices does T_k have? (Briefly justify your answer.) [4]

[4] As with part (b), start with a small tree and then repeatedly apply insights from part c or d. In the next two weeks you will see a more rigorous way to prove this kind of thing using induction.