Induction worksheet

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Fall 2020

TAKE-AWAYS

- *Induction* is a proof technique where to prove $\forall n \geq 0(P(n))$, you first prove $P(0)$ (the *base case*) and then prove $\forall k > 0((P(0) \wedge P(1) \wedge ... \wedge P(k-1)) \rightarrow P(k))$ (the *inductive case*)
- Sometimes you may need multiple base cases and/or a base case that isn't 0.
- Common errors in proofs by induction include omitting the base case, reversing the implication, writing an inductive step that fails for certain values, and using a $P(n)$ that isn't a predicate.

Problem 1. Solve Problem 5.16 parts (a)-(f) from our MCS textbook.

Problem 2. Solve Problem 5.23 from our MCS textbook.

Problem 3. Recall that one of de Morgan's Laws states $\neg(x \vee y) \equiv (\neg x) \wedge (\neg y)$. Fill in the template below to prove the following *generalized* de Morgan's Law:

For any $n \geq 2$, $\neg(x_1 \vee x_2 \vee ... \vee x_n) \equiv (\neg x_1) \wedge (\neg x_2) \wedge ... \wedge (\neg x_n)$

Proof. Define $P(n)$ to be "______". We proceed by induction on n.

Base case: We need to prove $P($______$)$. It's true because ______
Inductive case: Fix k>______, and assume as our inductive hypothesis that P holds for each i from ______ to $k-1$. Now we need to prove $P(k)$, which we do as follows: ______.

Induction complete. □

Problem 4. Be careful of things you find on the internet! The first relevant result of an "induction examples" Google search (`www.mathsisfun.com/algebra/mathematical-induction.html`) uses a different template than ours - instead of using $P(0) \wedge P(1) \wedge ... \wedge P(k-1)$ to prove $P(k)$ in the inductive step, they're using $P(k)$ to prove $P(k+1)$.[1] However even if they had been written with our template, every proof on that page would still have a problem:

a) What rather minor issue does the first proof (that $3^n - 1$ is always a multiple of 2) have?[2]

[1] In this class, we only use what they would call "strong induction".

[2] Hint: What's $P(n)$?

b) What much more serious error do all the other proofs have?[3]

c) Prove by induction that $\forall n \geq 1 (1 + 3 + 5 + ... + (2n - 1) = n^2)$. *(You may use the flawed proof on that website for guidance, but don't repeat their flaw, and make sure you use our template instead of theirs.)*

[3] Hint: It's one of the 4 common errors discussed in the "Bogus induction proofs" section of the notes.

Problem 5. The first two Fibonacci numbers are defined to be $F_0 = 0$ and $F_1 = 1$, and each subsequent number is the sum of the previous two, i.e. $F_2 = F_1 + F_0$, and in general $\forall n > 1$, $F_n = F_{n-1} + F_{n-2}$. So the first several are $0, 1, 1, 2, 3, 5, 8, 13....$ Prove the following:

$$\forall n \geq 1 (F_n \geq (3/2)^{n-2})$$