Worksheet on Pigeon Hole Principle and Principle of Inclusion-Exclusion

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Lecture Summary

- The pigeon hole principle says that if $|A| > |B|$ then for any function $f : A \to B$ there are $a, b \in A$ such that $f(a) = f(b)$.
- The generalized pigeon hole principle is as follows. Let A be a set and B be an n-element set (say) $\{b_1, b_2, \ldots b_n\}$. Let $q_1, \ldots q_n$ be n natural numbers such that

  $$|A| > q_1 + q_2 + \cdots + q_n.$$

  For any function $f : A \to B$ there is an $i \in \{1, 2, \ldots n\}$ such that $|\{a \in A \mid f(a) = b_i\}| > q_i$.
- Observe that the (basic) pigeon hole principle is a special case of the generalized pigeon hole principle, where each $q_i = 1$.
- Another special case of the generalized pigeon hole principle is as follows. If $|A| > k|B|$ then for any function $f : A \to B$ there are $k+1$ elements $a_1, a_2, \ldots a_{k+1} \in A$ such that $f(a_i) = f(a_j)$ for any $i, j \in \{1, 2, \ldots k+1\}$.
- The principle of inclusion-exclusion says that for any sets $S_1, S_2, \ldots S_n$,

  $$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots n\}} (-1)^{|I|+1} \left|\bigcap_{i \in I} S_i\right|$$
- When $n = 2$ or $n = 3$, the principle of inclusion-exclusion specializes to the following equations.

  $$\begin{aligned} |A \cup B| = {} & |A| + |B| - |A \cap B| \\ |A \cup B \cup C| = {} & |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C| \end{aligned}$$

Problem 1. Prove that any subset $A \subseteq \{1, 2, \ldots 9\}$ of size 6, must contain a pair of numbers whose sum is 10.

Problem 2. In a group of n people, prove that there are two people with the same number of friends.

Problem 3. For any sequence of integers $a_1, a_2, \dots a_n$, prove that there is some "consecutive sum" that is divisible by n. That is, prove that there are indicies $0 \leq i < j \leq n$ such that $n|(a_{i+1} + a_{i+2} + \cdots + a_j)$.

Problem 4. Let S be the set of permutations of $\{0, 1, 2, \dots 9\}$ such that either 7 and 3, or 1 and 7 appear consecutively (in that order). For example, $0, 1, 2, 3, 4, 5, 6, 7, 8, 9 \notin S$ (as neither 1.7 nor 7, 3 appear), $0, 2, 3, 4, 5, 6, 8, 9, 7, 1 \notin S$ (the 7, 1 at the end does not count as they are in the flipped order), $0, 2, 3, 4, 5, 6, 8, 9, 1, 7 \in S$ (because of the 1, 7 at the end), $1, 7, 3, 0, 2, 4, 5, 6, 8, 9 \in S$ (because of either 1 followed by 7 or 7 followed by 3). What is $|S|$?

Problem 5. How many ways can we place 4 distinct letters in 4 different pre-addressed envelopes so that no letter is placed the correct envelope?