Worksheet on Proofs

Benjamin Cosman, Patrick Lin and Mahesh Viswanathan

Fall 2020

Definition 1. An integer $n \in \mathbb{Z}$ is *even* if there is an integer k such that $n = 2k$. An integer n is *odd* if there is an integer k such that $n = 2k + 1$.

Definition 2. A real number $r \in \mathbb{R}$ is a *rational* number if there are integers $a, b \in \mathbb{Z}$ such that $b \neq 0$ and $r = \frac{a}{b}$. If r is a rational number, one can assume that the integers a, b are in lowest terms, i.e., they do not share any common factors.

A real number r is *irrational* if r is not rational.

Definition 3. For any real number $x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition 4. For positive real numbers x and y, $\log_x y$ is the (unique) real number z such that $x^z = y$.

Problem 1. Prove that for any real number $r \in \mathbb{R}$, if $2r$ is rational then r is rational.

Problem 2. Prove that for any real numbers $x, y \in \mathbb{R}$ with $x \neq 0$, if x and $\frac{y+1}{3}$ are rational then $\frac{1}{x} + y$ is rational.

Problem 3. Prove that for any integer $n \in \mathbb{Z}$, if $3n + 2$ is odd then n is odd.

Problem 4. Prove that for any integer n, $n \leq n^2$.

Problem 5. Prove the following observations.

(a) $\log_2 3$ is irrational.

(b) Given that $\log_2 3$ is irrational, $2\log_2 3$ is irrational. *Hint:* Can you use Problem 1?

(c) There are irrational numbers x, y such that x^y is rational. *Hint:* Use the previous part and recall that $\sqrt{2}$ is irrational.