CS 173, Spring 19 Examlet 10, colored

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{p}{p+1} \leq \frac{n^2}{n+1}$ for all positive integers n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{p}{p+1} = \frac{1}{2}$ and $\frac{n^2}{n+1} = \frac{1}{2}$. So the claim holds at $n = 1$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{p}{p+1} \leq \frac{n^2}{n+1}$ for $n = 1, \ldots, k$.

Inductive Step:

First, let's prove the following lemma: $\frac{k^2}{k+1} \leq \frac{k(k+1)}{k+2}$.

Proof of lemma: Notice that $k(k+2) = k^2 + 2k \leq k^2 + 2k + 1 = (k+1)^2$. So $k(k+2) \leq (k+1)^2$. So (since k is positive) $\frac{k}{k+1} \leq \frac{k+1}{k+2}$. So $\frac{k^2}{k+1} \leq \frac{k(k+1)}{k+2}$.

Now by the inductive hypothesis $\sum_{p=1}^{k} \frac{p}{p+1} \leq \frac{k^2}{k+1}$ So

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{p}{p+1} &= \frac{k+1}{k+2} + \sum_{p=1}^{k} \frac{p}{p+1} \\
&\leq \frac{k+1}{k+2} + \frac{k^2}{k+1} \leq \frac{k+1}{k+2} + \frac{k(k+1)}{k+2} \\
&= \frac{k^2+2k+1}{k+2} = \frac{(k+1)^2}{k+2}
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{p}{p+1} \leq \frac{(k+1)^2}{k+2}$ which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $(2n)!^2 < (4n)!$ for all positive integers.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $(2n)!^2 = (2!)^2 = 2^2 = 4$ And $(4n)! = 4! = 24$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $(2n)!^2 < (4n)!$ for $n = 1, 2, \ldots, k$.

Inductive Step: At $n = k + 1$, we have

$(2(k + 1))!^2 = (2k + 2)!^2 = [(2k + 2)(2k + 1)(2k!)]^2 = (2k + 2)(2k + 2)(2k + 1)(2k + 1)(2k)!^2$

Also $(4(k + 1))! = (4k + 4)! = (4k + 4)(4k + 3)(4k + 2)(4k + 1)(4k)!$

Also notice that $(2k + 2)(2k + 2)(2k + 1)(2k + 1) < (4k + 4)(4k + 3)(4k + 2)(4k + 1)$ because each of the four terms on the left is smaller than the four terms on the right.

From the inductive hypothesis, we know that $(2k)!^2 < (4k)!$.

Putting this all together, we get

$$\begin{aligned}
(2(k+1))!^2 &= (2k+2)(2k+2)(2k+1)(2k+1)(2k)!^2 \\
&< (2k+2)(2k+2)(2k+1)(2k+1)(4k)! \\
&< (4k+4)(4k+3)(4k+2)(4k+1)(4k)! \\
&= (4(k+1))!
\end{aligned}$$

So $(2(k + 1))!^2 < (4(k + 1))!$, which is what we needed to prove.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any reals $a_1, \ldots, a_n$ between 0 and 1 (inclusive)

$$\prod_{p=1}^{n}(1-a_p) \geq 1 - \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1-a_p) = 1-a_1$ and $1-\sum_{p=1}^{n} a_p = 1-a_1$ so $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1-a_p) \geq 1-\sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any real numbers $a_1, \ldots, a_n$ between 0 and 1 (inclusive).

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be real numbers between 0 and 1 (inclusive). By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1-a_p) \geq 1-\sum_{p=1}^{k} a_p$. Since $(1-a_{k+1})$ is positive, this means that $(1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p)$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1-a_p) &= (1-a_{k+1})\prod_{p=1}^{k}(1-a_p) \\
&\geq (1-a_{k+1})(1-\sum_{p=1}^{k} a_p) \;=\; 1-a_{k+1}+a_{k+1}\sum_{p=1}^{k} a_p - \sum_{p=1}^{k} a_p \\
&\geq 1-a_{k+1}-\sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1-\sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1-a_p) \geq 1-\sum_{p=1}^{k+1} a_p$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\displaystyle\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} > 0$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} = 1 > 0$. So the claim holds. At $n = 2$, $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} = 1 - 1/2 = 1/2 > 0$. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{p=1}^{n} \frac{(-1)^{p-1}}{p} > 0$ for $n = 1, 2, \ldots, k$.

Inductive Step: There are two cases:

Case 1) k is even.

$\sum_{p=1}^{k+1} \frac{(-1)^{k-1}}{k} = \frac{(-1)^k}{k+1} + \sum_{p=1}^{k} \frac{(-1)^{p-1}}{p}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k} \frac{(-1)^{p-1}}{p}$ is positive. Since k is even, we know that $\frac{(-1)^k}{k+1}$ is positive. Since $\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p}$ is the sum of two positive numbers, it must be positive.

Case 2) k is odd. Then remove two terms from the summation:

$\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p} = \frac{(-1)^{k-1}}{k} + \frac{(-1)^k}{k+1} + \sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{p}$.

From the inductive hypothesis, we know that $\sum_{p=1}^{k-1} \frac{(-1)^{p-1}}{p}$ is positive. Since k is odd, $\frac{(-1)^{k-1}}{k} + \frac{(-1)^k}{k+1} = \frac{1}{k} + \frac{-1}{k+1} = \frac{1}{k} - \frac{1}{k+1}$. Since $\frac{1}{k}$ is larger than $\frac{1}{k+1}$, $\frac{1}{k} - \frac{1}{k+1}$ is positive. Since $\sum_{p=1}^{k+1} \frac{(-1)^{p-1}}{p}$ is the sum of two positive numbers, it must be positive.

In both cases, we have show that $\sum_{p=1}^{k+1} \frac{(-1)^k}{k+1} > 0$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\frac{(2n)!}{n!n!} > 2^n$, for all integers $n \geq 2$

Solution:

Proof by induction on n.

Base Case(s): At $n = 2$, $\frac{(2n)!}{n!n!} = \frac{4!}{2!2!} = \frac{24}{4} = 6 > 4 = 2^n$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\frac{(2n)!}{n!n!} > 2^n$, for $n = 2, \ldots, k$.

Inductive Step: By the inductive hypothesis, $\frac{(2k)!}{k!k!} > 2^k$.

Also notice that $2k + 1 > k + 1$ because $k \geq 0$. So $\frac{2k+1}{k+1} > 1$.

Then we can compute

$$\begin{aligned} \frac{(2(k+1))!}{(k+1)!(k+1)!} &= \frac{(2k+2)(2k+1)(2k)!}{(k+1)k!(k+1)k!} = \frac{(2k+2)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} \\ &> \frac{(2k+2)(2k+1)}{(k+1)^2}2^k \\ &= \frac{(k+1)(2k+1)}{(k+1)^2}2^{k+1} = \frac{2k+1}{k+1}2^{k+1} > 2^{k+1} \end{aligned}$$

So $\frac{(2(k+1))!}{(k+1)!(k+1)!} > 2^{k+1}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: For any natural number n and any real number x, where $0 < x < 1$, $(1-x)^n \geq 1-nx$.

Let x be a real number, where $0 < x < 1$.

Solution:

Proof by induction on n.

Base Case(s): At $n = 0$, $(1-x)^n = (1-x)^0 = 1$ and $1 - nx = 1 + 0 = 1$. So $(1-x)^n \geq 1 - nx$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $(1-x)^n \geq 1 - nx$ for any natural number $n \leq k$, where k is a natural number.

Inductive Step: By the inductive hypothesis $(1-x)^k \geq 1 - kx$. Notice that $(1-x)$ is positive since $0 < x < 1$. So $(1-x)^{k+1} \geq (1-x)(1-kx)$.

But $(1-x)(1-kx) = 1 - x - kx + kx^2 = 1 - (1+k)x + kx^2$.

And $1 - (1+k)x + kx^2 \geq 1 - (1+k)x$ because kx^2 is non-negative.

So $(1-x)^{k+1} \geq (1-x)(1-kx) \geq 1-(1+k)x$, and therefore $(1-x)^{k+1} \geq 1-(1+k)x$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{p=1}^{n} \frac{1}{p} \ \leq \ \frac{n}{2} + 1$, for any positive integer n.

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{p} = 1$. Also $\frac{n}{2} + 1 = 1.5$, which is larger. So the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{1}{p} \ \leq \ \frac{n}{2} + 1$, for $n = 1, \ldots, k$.

Inductive Step: In particular, by the inductive hypothesis, $\sum_{p=1}^{k} \frac{1}{p} \ \leq \ \frac{k}{2} + 1$. Also notice that k is positive, so $k + 1 \geq 2$, and therefore $\frac{1}{k+1} \leq \frac{1}{2}$. Thus $\frac{1}{k+1} - \frac{1}{2} \leq 0$. So

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{p} &= \frac{1}{k+1} + \sum_{p=1}^{k} \frac{1}{p} \ \leq \ \frac{1}{k+1} + \frac{k}{2} + 1 \\
&= \left(\frac{k+1}{2} - \frac{k+1}{2}\right) + \left(\frac{1}{k+1} + \frac{k}{2} + 1\right) && \text{based on backwards scratch work} \\
&= \left(\frac{k+1}{2} + 1\right) + \frac{1}{k+1} + \left(\frac{k}{2} - \frac{k+1}{2}\right) && \text{rearrange terms} \\
&= \left(\frac{k+1}{2} + 1\right) + \frac{1}{k+1} - \frac{1}{2} \\
&\leq \frac{k+1}{2} + 1 && \text{because } \frac{1}{k+1} - \frac{1}{2} \leq 0
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{1}{p} \leq \frac{k+1}{2} + 1$, which is what we needed to show.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

Claim: For any positive integer n and any positive reals $a_1, \ldots, a_n$,

$$\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$$

Solution:

Proof by induction on n.

Base Case(s): At $n = 1$, $\prod_{p=1}^{n}(1+a_p) = 1+a_1$ and $1+\sum_{p=1}^{n} a_p = 1+a_1$ so $\prod_{p=1}^{n}(1+a_p) \geq 1+\sum_{p=1}^{n} a_p$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=1}^{n}(1+a_p) \geq 1 + \sum_{p=1}^{n} a_p$ for $n = 1, \ldots, k$ and any positive real numbers $a_1, \ldots, a_n$.

Inductive Step: Let $a_1, \ldots, a_{k+1}$ be positive real numbers. By the inductive hypothesis, we know that $\prod_{p=1}^{k}(1+a_p) \geq 1 + \sum_{p=1}^{k} a_p$. Then we have

$$\begin{aligned}
\prod_{p=1}^{k+1}(1+a_p) &= (1+a_{k+1})\prod_{p=1}^{k}(1+a_p) \\
&\geq (1+a_{k+1})(1+\sum_{p=1}^{k} a_p) \;=\; 1 + a_{k+1} + a_{k+1}\sum_{p=1}^{k} a_p + \sum_{p=1}^{k} a_p \\
&\geq 1 + a_{k+1} + \sum_{p=1}^{k} a_p \quad \text{because all values } a_p \text{ are positive} \\
&= 1 + \sum_{p=1}^{k+1} a_p
\end{aligned}$$

So $\prod_{p=1}^{k+1}(1+a_p) \geq 1 + \sum_{p=1}^{k+1} a_p$, which is what we needed to show.