CS 173, Spring 19 Examlet 4, colored

Name:____

NetID:________________________ **Lecture: A B**

Discussion: Thursday Friday 9 10 11 12 1 2 3 4 5 6

Let's define a relation T between natural numbers follows:

aTb if and only if $a = b + 2k$, where k is a natural number

Working directly from this definition, prove that T is antisymmetric.

Solution: Let a and b be natural numbers and suppose that aTb and bTa.

By the definition of T, this means that $a = b+2k$ and $b = a+2j$, where k and j are natural numbers.

Substituting one equation into the other, we get $a = (a + 2j) + 2k = a + 2(j + k)$. So $2(j + k) = 0$. So $j + k = 0$.

Notice that j and k are both non-negative. So $j + k = 0$ implies that $j = k = 0$.

So $a = b$, which is what we needed to show.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

A closed interval of the real line can be represented as a pair (c, r), where c is the center of the interval and r is its radius. Let $X = \{(c, r) \mid c, r \in \mathbb{R}, r \geq 0\}$ be the set of closed intervals represented this way.

Now, let's define the interval containment $\preceq$ on X as follows

$(c, r) \preceq (d, q)$ if and only if $r \leq q$ and $|c - d| + r \leq q$.

Prove that $\preceq$ is transitive.

Solution: Let (c, r), (d, q), and (f, s) be elements of X. Suppose that $(c, r) \preceq (d, q)$ and $(d, q) \preceq (f, s)$. By the definition of $\preceq$, this means that $r \leq q$ and $|c - d| + r \leq q$ and $q \leq s$ and $|d - f| + q \leq s$. So $r \leq s$. Also, $|c-d|+r+|d-f|+q \leq q+s$, which implies that $|c-f|+r \leq |c-d|+|d-f|+r \leq s$. So $(c, r) \prec (f, s)$, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Suppose that T is a relation on the integers which is transitive. Let's define a relation R on the integers as follows:

xRy if and only if there is an integer k such that xTk and kTy.

Prove that R is transitive.

Solution: Let a, b and c be integers. Suppose that aRb and bRc.

By the definition of R, aRb means that there is an integer k such that aTk and kTb. Since T is known to be transitive, this implies that aTb.

Similarly bRc means that there is an integer j such that bTj and jTc. And (because T is transitive), therefore bTc.

We now know that aTb and bTc, where b is an integer. So, by the definition of R, aRc, which is what we needed to show.

CS 173, Spring 19 Examlet 4, colored

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let T be the relation defined on $\mathbb{Z}^2$ by

$(x, y)T(p, q)$ if and only if $x < p$ or $(x = p$ and $y \leq q)$

Prove that T is antisymmetric.

Solution:

Let (x, y) and (p, q) be pairs of integers. Suppose that $(x, y)T(p, q)$ and $(p, q)T(x, y)$. By the definition of T $(x, y)T(p, q)$ means that $x < p$ or $(x = p$ and $y \leq q)$. Similarly, $(p, q)T(x, y)$ means that $p < x$ or $(p = x$ and $q \leq y)$.

There are four cases:

Case 1: $x < p$ and $p < x$. This is impossible.

Case 2: $x < p$ and $p = x$ and $q \leq y$. Also impossible.

Case 3: $p < x$ and $x = p$ and $y \leq q$. Impossible as well.

Case 4: $x = p$ and $y \leq q$ and $p = x$ and $q \leq y$. Since $y \leq q$ and $q \leq y$, $x = y$. So we have $(x, y) = (p, q)$.

$(x, y) = (p, q)$ is true, which is what we needed to show.

Name:____

NetID:____________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Define the relation $\sim$ on $\mathbb{Z}$ by

$x \sim y$ if and only if $5 \mid (3x + 7y)$

Working directly from the definition of divides, prove that $\sim$ is transitive.

Solution: Let x, y, and z be integers. Suppose that $x \sim y$ and $y \sim z$.

By the definition of $\sim$, $5 \mid (3x + 7y)$ and $5 \mid (3y + 7z)$. So $3x + 7y = 5m$ and $3y + 7z = 5n$, for some integers m and n.

Adding these two equations together, we get $3x+7y+3y+7z = 5m+5n$. So $3x+10y+7z = 5(m+n)$. So $3x + 7z = 5(m + n - 2y)$.

$m + n - 2y$ is an integer, since m, n and y are integers. So this means that $5 \mid 3x + 7z$ and therefore $x \sim z$, which is what we needed to show.

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \{(x, y, z) \in \mathbb{Z}^3 : x \le y \le z\}$. Let's define a relation R on A as follows:

$(a, b, c)R(x, y, z)$ if and only if $a \le x$ and $z \le b$.

Working directly from this definition, prove that R is antisymmetric.

Solution: Let (x, y, z) and (a, b, c) be elements of A. Suppose that $(x, y, z)R(a, b, c)$ and $(a, b, c)R(x, y, z)$.

By the definition of R, $(a, b, c)R(x, y, z)$ implies that $a \le x$ and $z \le b$. Similarly, $(x, y, z)R(a, b, c)$ implies that $x \le a$ and $c \le y$.

We have $a \le x$ and $x \le a$, so $x = a$.

We also have $z \le b$ and $c \le y$. But notice that we also know that $x \le y \le z$ and $a \le b \le c$ from the definition of A. Combining these inequlities, we have

$$b \le c \le y \le z \le b$$

So $b = c = y = z$.

So $(x, y, z) = (a, b, c)$, which is what we needed to prove.

Name:____

NetID:______________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \mathbb{Z}^+ \times \mathbb{Z}^+$, i.e. pairs of positive integers.

Define a relation $\gg$ on A as follows:

$$(x, y) \gg (p, q) \text{ if and only if there exists an integer } n \geq 1 \text{ such that } (x, y) = (np, nq).$$

Prove that $\gg$ is antisymmetric.

Solution: Let (x, y) and (p, q) be pairs of natural numbers and suppose that $(x, y) \gg (p, q)$ and $(p, q) \gg (x, y)$.

By the definition of $\gg$, $(x, y) = (np, nq)$ and $(p, q) = m(x, y)$, for some positive integers m and n. So $x = np$, $y = nq$, $p = mx$ and $q = my$.

Combining these equations, we get $x = n(mx) = (nm)x$ and $y = n(my) = (nm)y$. So $nm = 1$. But this means that $n = m = 1$ since n and m are positive integers. So $x = p$ and $y = q$. So $(x, y) = (p, q)$, which is what we needed to show.

CS 173, Spring 19 Examlet 4, colored

Name:____

NetID:________________________ **Lecture:** **A** **B**

Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

Let $A = \{(x, y, z) \in \mathbb{Z}^3 : x \leq y \leq z\}$. Let's define a relation R on A as follows:

$(a, b, c)R(x, y, z)$ if and only if $a \leq x$ and $z \leq b$.

Working directly from this definition, prove that R is transitive.

Solution: Let (x, y, z), (a, b, c), and (p, q, r) be elements of A. Suppose that $(x, y, z)R(a, b, c)$ and $(a, b, c)R(p, q, r)$.

By the definition of R, $(x, y, z)R(a, b, c)$ implies that $x \leq a$ and $c \leq y$. Similarly $(a, b, c)R(p, q, r)$ implies that $a \leq p$ and $r \leq b$.

So have $x \leq a$ and $a \leq p$, so $x \leq p$.

We also have $c \leq y$ and $r \leq b$. Notice that $a \leq b \leq c$ by the definitino of the set A. So we have $r \leq b \leq c \leq y$, and therefore $r \leq y$

Since $x \leq p$ and $r \leq y$, $(x, y, z)R(p, q, r)$, which is what we needed to show.