CS 173, Spring 19 Examlet 5, colored

Name:____

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Discussion: **Thursday** **Friday** **9** **10** **11** **12** **1** **2** **3** **4** **5** **6**

1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x - 7)f(y)$. Prove that g is onto.

   Solution: Suppose that n is an integer.

   Since f is onto, there is an integer p such that $f(p) = 1$. Let $x = p + 7$. Then $f(x - 7) = f(p) = 1$.

   Also since f is onto, there is a natural number y such that $f(y) = n$.

   Now consider the pair (x, y). $g(x, y) = f(x - 7)f(y) = 1 \cdot n = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "onto." You must use explicit quantifiers. Do not assume the reader knows what the image of the function is.

   Solution: For every element y in M, there is an element x in C such that $g(x) = y$.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is one-to-one. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}^2$ by $g(x, y) = (2f(x) + f(y), f(x) - f(y))$. Prove that g is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x, y) and (p, q) be elements of $\mathbb{Z}^2$ and suppose that $g(x, y) = g(p, q)$.

   By the definition of h, this means that $(2f(x) + f(y), f(x) - f(y)) = (2f(p) + f(q), f(p) - f(q))$. So $2f(x) + f(y) = 2f(p) + f(q)$ and $f(x) - f(y) = f(p) - f(q)$.

   Adding these two equations, we get $3f(x) = 3f(p)$. So $f(x) = f(p)$. Since f is one-to-one, this means that $x = p$.

   Subtracting twice the second equation from the first, we get $-3f(y) = -3f(q)$. So $f(y) = f(q)$. Since f is one-to-one, this means that $y = q$.

   Since $x = p$ and $y = q$, $(x, y) = (p, q)$, which is what we needed to show.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : C \to M$ to be "one-to-one." You must use explicit quantifiers; do not use words like "unique".

   Solution: For every elements x and y in C, if $g(x) = g(y)$, then $x = y$

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1. (10 points) Suppose that A and B are sets. Suppose that $f : B \to A$ and $g : A \to B$ are functions such that $f(g(x)) = x$ for every $x \in A$. Prove that f is onto.

   Solution: Let m be an element of A. We need to find a pre-image for m.

   Consider $n = g(m)$. n is an element of B. Furthermore, since $f(g(x)) = x$ for every $x \in A$, we have $f(n) = f(g(m)) = m$.

   So n is a pre-image of m.

   Since we can find a pre-image for an arbitrarily chosen element of A, f is onto.

2. (5 points) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto. Your answer must include a specific formula.

   Solution: Let $f(n) = n + 1$. Then f is one-to-one, but 0 isn't in the image of f.

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1. (10 points) Suppose that $f : \mathbb{Z} \to \mathbb{Z}$ is onto. Let's define $g : \mathbb{Z}^2 \to \mathbb{Z}$ by $g(x, y) = f(x) + 2f(y) - 6$. Prove that g is onto.

   Solution: Let n be an arbitrary integer.

   Since f is onto, there is an integer input value y such that $f(y) = 3$. Similarly, there is an integer input value x such that $f(x) = n$.

   Now, consider (x, y). $g(x, y) = f(x) + 2f(y) - 6 = n + 2 \cdot 3 - 6 = n$. So (x, y) is a pre-image for n, which is what we needed to find.

2. (5 points) $A = \{0, 2, 4, 6, 8, 10, 12, \ldots\}$, i.e. the even integers starting with 0.

   $B = \{1, 4, 9, 16, 25, 36, 49, \ldots\}$, i.e. perfect squares starting with 1.

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = (\frac{n}{2} + 1)^2$

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1. (10 points) Suppose that $f : A \to B$ and $g : B \to C$ are one-to-one. Prove that $g \circ f$ is one-to-one.

   Solution: Let x and y be elements of A and suppose that $g \circ f(x) = g \circ f(y)$. That is $g(f(x)) = g(f(y))$. Since g is one-to-one, this implies that $f(x) = f(y)$. Since f is one-to-one, this implies that $x = y$.

   We've shown that $g \circ f(x) = g \circ f(y)$ implies $x = y$ for any x and y in A. So $g \circ f$ is one-to-one.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x \leq y$ then $g(x) \leq g(y)$.

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1. (10 points) Let P be the set of pairs of positive integers. Suppose that $f : P \rightarrow \mathbb{R}^2$ is defined by $f(x,y) = (\frac{x}{y}, x+y)$. Prove that f is one-to-one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let (x,y) and (p,q) be elements of P, i.e. pairs of positive integers. Suppose that $f(x,y) = f(p,q)$.

   By the definition of f, this means that $(\frac{x}{y}, x+y) = (\frac{p}{q}, p+q)$. So $\frac{x}{y} = \frac{p}{q}$ and $x+y = p+q$.

   Since $\frac{x}{y} = \frac{p}{q}$, $x = \frac{py}{q}$. Substituting this into $x+y = p+q$ gives us $\frac{py}{q} + y = p+q$. So $\frac{py+yq}{q} = p+q$. I.e. $\frac{y(p+q)}{q} = p+q$. So $\frac{y}{q} = 1$, and therefore $y = q$.

   Substituting $y = q$ into $x+y = p+q$ gives us $x+y = p+y$, so $x = p$.

   Therefore $(x,y) = (p,q)$, which is what we needed to prove.

2. (5 points) Give an example of a function $f : \mathbb{N} \rightarrow \mathbb{N}$ which is onto but not one-to-one. Your answer must include a specific formula.

   Solution: Let $f(n) = \lfloor n/2 \rfloor$ Then f is onto. But f isn't one-to-one because (for example) both 0 and 1 are mapped onto 0.

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1. (10 points) Suppose that $f : \mathbb{Z}^2 \to \mathbb{Z}$ is defined by $f(x, y) = 3x + 5y$. Prove that f is onto.

   Solution: Let p be an integer. We need to find a pre-image for p.

   Consider $m = (-3p, 2p)$.

   m is an element of $\mathbb{Z}^2$. We can compute

   $$f(m) = f(-3p, 2p) = 3(-3p) + 5(2p) = -9p + 10p = p$$

   So m is a pre-image of p.

   Since we can find a pre-image for an arbitrarily chosen integer, f is onto.

2. (5 points) $A = \{0, 1, 4, 9, 16, 25, 36, \ldots\}$, i.e. perfect squares starting with 0.

   $B = \{2, 4, 6, 8, 10, 12, 14, \ldots\}$, i.e. the even integers starting with 2.

   Give a specific formula for a bijection $f : A \to B$. (You do not need to prove that it is a bijection.)

   Solution: $f(n) = 2(\sqrt{n} + 1)$

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1. (10 points) Let's define the function $f : \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \dfrac{4x-1}{2x+5}$. ($\mathbb{R}^+$ is the positive reals.) Prove that f is one to one. You must work directly from the definition of one-to-one. Do not use any facts about (for example) derivatives or the behavior of increasing functions.

   Solution: Let x and y be positive reals. Suppose that $f(x) = f(y)$. By the definition of f, this means that $\frac{4x-1}{2x+5} = \frac{4y-1}{2y+5}$.

   Multiplying by the two denominators gives us $(4x-1)(2y+5) = (4y-1)2x+5$. That is $8xy - 2y + 20x - 5 = 8xy - 2x + 20y - 5$. So $-2y + 20x = -2x + 20y$. So $22x = 22y$. And therefore $x = y$, which is what we needed to prove.

2. (5 points) Using precise mathematical words and notation, define what it means for a function $g : \mathbb{R} \to \mathbb{R}$ to be "strictly increasing." You must use explicit quantifiers.

   Solution: For all x and y in $\mathbb{R}$, if $x < y$ then $g(x) < g(y)$.