CS 173, Spring 19 Examlet 7, colored

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(15 points) Working directly from the definition of divides, use (strong) induction to prove the following claim:

Claim: $n^3 + 5n$ is divisible by 6, for all positive integers n.

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $n^3 + 5n = 6$, which is clearly divisible by 6.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that n^3+5n is divisible by 6, for $n = 1, 2, \dots, k$.

Rest of the inductive step: Notice that

$$(k+1)^3+5(k+1) = (k^3+3k^2+3k+1)+(5k+5) = (k^3+5k)+(3k^2+3k)+6 = (k^3+5k)+3k(k+1)+6$$

$(k^3 + 5k)$ is divisible by 6 by the inductive hypothesis. $3k(k + 1)$ is divisible by 6 because one of k and (k+1) must be even. 6 is obviously divisible by 6. Since $(k+1)^3 + 5(k+1)$ is the sum of these three terms, it must also be divisible by 6, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\displaystyle\sum_{p=1}^{n} \log(p^2) = 2\log(n!)$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} \log(p^2) = \log(1^2) = \log 1 = 0$. Also $2\log(n!) = 2\log 1 = 2 \cdot 0 = 0$. So the two are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\displaystyle\sum_{p=1}^{n} \log(p^2) = 2\log(n!)$. for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, $\displaystyle\sum_{p=1}^{k} \log(p^2) = 2\log(k!)$. Then

$$\begin{aligned}
\sum_{p=1}^{k+1} \log(p^2) &= \log((k+1)^2) + \sum_{p=1}^{k} \log(p^2) \;=\; 2\log(k+1) + \sum_{p=1}^{k} \log(p^2) \\
&= 2\log(k+1) + 2\log(k!) \;=\; 2(\log(k+1) + \log(k!)) \\
&= 2\log((k+1)k!) \;=\; 2\log((k+1)!)
\end{aligned}$$

So $\displaystyle\sum_{p=1}^{k+1} \log(p^2) = 2\log((k+1)!)$, which is what we needed to show.

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(15 points) Let A be a constant integer. Use (strong) induction to prove the following claim. Remember that $0! = 1$.

Claim: For any integer $n \geq A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Solution: Proof by induction on n.

Base case(s): At $n = A$, $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{A!}{A!0!} = 1 = \frac{(A+1)!}{(A+1)!0!} = \frac{(n+1)!}{(A+1)!(n-A)!}$

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=A}^{n} \frac{p!}{A!(p-A)!} = \frac{(n+1)!}{(A+1)!(n-A)!}$ is true for all $n = A, \ldots, k$.

Rest of the inductive step:

In particular, $\sum_{p=A}^{k} \frac{p!}{A!(p-A)!} = \frac{(k+1)!}{(A+1)!(k-A)!}$. So then

$$\begin{aligned}
\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} &= \sum_{p=A}^{k} \frac{p!}{A!(p-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1)!}{(A+1)!(k-A)!} + \frac{(k+1)!}{A!(k+1-A)!} \\
&= \frac{(k+1-A)(k+1)!}{(A+1)!(k+1-A)!} + \frac{(A+1)(k+1)!}{(A+1)!(k+1-A)!} \\
&= \frac{(k+2)(k+1)!}{(A+1)!(k+1-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}
\end{aligned}$$

So $\sum_{p=A}^{k+1} \frac{p!}{A!(p-A)!} = \frac{(k+2)!}{(A+1)!(k+1-A)!}$, which is what we needed to show.

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(15 points) The operator $\prod$ is like $\sum$ except that it multiplies its terms rather than adding them. So e.g. $\prod_{p=3}^{5}(p+1) = 4 \cdot 5 \cdot 6$. Use (strong) induction to prove the following claim:

$\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = \frac{n+1}{2n}$ for any integer $n \geq 2$.

Solution: Proof by induction on n.

Base case(s): At $n = 2$, $\prod_{p=2}^{n}(1 - \frac{1}{p^2}) = (1 - \frac{1}{4}) = \frac{3}{4}$ and $\frac{n+1}{2n} = \frac{3}{4}$, so the claim holds.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\prod_{p=2}^{n}(1-\frac{1}{p^2}) = \frac{n+1}{2n}$ for $n = 2, \ldots, k$

Rest of the inductive step: In particular, from the inductive hypothesis $\prod_{p=2}^{k}(1 - \frac{1}{p^2}) = \frac{k+1}{2k}$.

So

$$\begin{aligned}
\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) &= (\prod_{p=2}^{k}(1 - \frac{1}{p^2}))(1 - \frac{1}{(k+1)^2}) \\
&= (\frac{k+1}{2k})(1 - \frac{1}{(k+1)^2}) = \frac{k+1}{2k} - \frac{k+1}{2k(k+1)^2} \\
&= \frac{k+1}{2k} - \frac{1}{2k(k+1)} = \frac{(k+1)^2}{2k(k+1)} - \frac{1}{2k(k+1)} \\
&= \frac{(k+1)^2 - 1)}{2k(k+1)} = \frac{k^2 + 2k)}{2k(k+1)} = \frac{k(k+2)}{2k(k+1)} = \frac{k+2}{2(k+1)}
\end{aligned}$$

So $\prod_{p=2}^{k+1}(1 - \frac{1}{p^2}) = \frac{k+2}{2(k+1)}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: for all natural numbers n, $\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\sum_{j=0}^{n} 2(-7)^j = 2$ and $\frac{1-(-7)^{n+1}}{4} = \frac{1-(-7)}{4} = 2$. So the claim holds at $n = 0$.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{j=0}^{n} 2(-7)^j = \frac{1-(-7)^{n+1}}{4}$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

In particular $\sum_{j=0}^{k} 2(-7)^j = \frac{1-(-7)^{k+1}}{4}$. So then

$$
\begin{aligned}
\sum_{j=0}^{k+1} 2(-7)^j &= \Big(\sum_{j=0}^{n} 2(-7)^j\Big) + 2(-7)^{k+1} \\
&= \frac{1-(-7)^{k+1}}{4} + 2(-7)^{k+1} = \frac{1-(-7)^{k+1}+8(-7)^{k+1}}{4} = \frac{1+7(-7)^{k+1}}{4} \\
&= \frac{1-(-7)^{k+2}}{4}
\end{aligned}
$$

So $\sum_{j=0}^{k+1} 2(-7)^j = \frac{1-(-7)^{k+2}}{4}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim.

Claim: For any positive integer n, $\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{n}$

Solution: Proof by induction on n.

Base case(s): At $n = 1$, $\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \frac{1}{\sqrt{1-1}+\sqrt{1}} = \frac{1}{\sqrt{0}+\sqrt{1}} = 1$ and $\sqrt{n} = \sqrt{1} = 1$. So the claim holds.

Inductive hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\sum_{p=1}^{n} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{n}$ for $n = 1, \ldots, k$.

Rest of the inductive step: In particular, $\sum_{p=1}^{k} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{k}$. So then

$$\begin{aligned}
\sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1}+\sqrt{p}} &= \frac{1}{\sqrt{k}+\sqrt{k+1}} + \sum_{p=1}^{k} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \frac{1}{\sqrt{k}+\sqrt{k+1}} + \sqrt{k} \\
&= \frac{\sqrt{k}-\sqrt{k+1}}{k-(k+1)} + \sqrt{k} = \frac{\sqrt{k}-\sqrt{k+1}}{-1} + \sqrt{k} \\
&= (\sqrt{k+1}-\sqrt{k}) + \sqrt{k} = \sqrt{k+1}
\end{aligned}$$

So $\sum_{p=1}^{k+1} \frac{1}{\sqrt{p-1}+\sqrt{p}} = \sqrt{k+1}$, which is what we needed to show.

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(15 points) Use (strong) induction to prove the following claim:

Claim: $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for all positive integers n.

Solution:

Proof by induction on n.

Base case(s): $n = 1$. At $n = 1$, $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{1(1+1)} = \frac{1}{2}$. Also $\frac{n}{n+1} = \frac{1}{2}$. So the two sides of the equation are equal.

Inductive hypothesis [Be specific, don't just refer to "the claim"]: Suppose that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$ for $n = 1, \ldots, k$ for some integer $k \geq 1$.

Rest of the inductive step:

Consider $\sum_{j=1}^{k+1} \frac{1}{j(j+1)}$. By removing the top term of the summation and then applying the inductive hypothesis, we get

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{1}{(k+1)(k+2)} + \frac{k}{k+1}.$$

Adding the two fractions together:

$$\frac{1}{(k+1)(k+2)} + \frac{k}{k+1} = \frac{1}{(k+1)(k+2)} + \frac{k(k+2)}{(k+1)(k+2)} = \frac{1+k(k+2)}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

So $\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \frac{k+1}{k+2}$, which is what we needed to show.

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(15 points) Use (strong) induction and the fact that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ to prove the following claim:

For all natural numbers n, $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$

Solution: Proof by induction on n.

Base case(s): At $n = 0$, $\left(\sum_{i=0}^{n} i\right)^2 = 0^2 = 0 = \sum_{i=0}^{n} i^3$. So the claim is true.

Inductive Hypothesis [Be specific, don't just refer to "the claim"]:

Suppose that $\left(\sum_{i=0}^{n} i\right)^2 = \sum_{i=0}^{n} i^3$ for $n = 0, 1, \ldots, k$.

Rest of the inductive step:

Starting with the lefthand side of the equation for $n = k + 1$, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = \left((k+1) + \sum_{i=0}^{k} i\right)^2 = (k+1)^2 + 2(k+1)\sum_{i=0}^{k} i + \left(\sum_{i=0}^{k} i\right)^2$$

By the inductive hypothesis $\left(\sum_{i=0}^{k} i\right)^2 = \sum_{i=0}^{k} i^3$. Substituting this and the fact we were told to assume, we get

$$\left(\sum_{i=0}^{k+1} i\right)^2 = (k+1)^2 + 2(k+1)\frac{k(k+1)}{2} + \sum_{i=0}^{k} i^3 = (k+1)^2 + k(k+1)^2 + \sum_{i=0}^{k} i^3 = (k+1)^3 + \sum_{i=0}^{k} i^3 = \sum_{i=0}^{k+1} i^3$$

So $\left(\sum_{i=0}^{k+1} i\right)^2 = \sum_{i=0}^{k+1} i^3$ which is what we needed to show.