Data Structures
Minimum Spanning Tree

CS 225
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Department of Computer Science

Learning Objectives

Review graph traversal algorithms

Introduce the minimum spanning tree (with weights)

Introduce Kruskal's / Prim's MST Algorithms

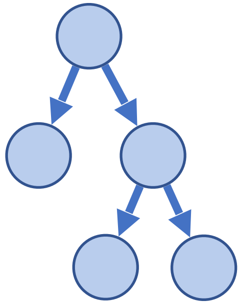
Begin discussing implementation of Kruskal's

Graph Traversals

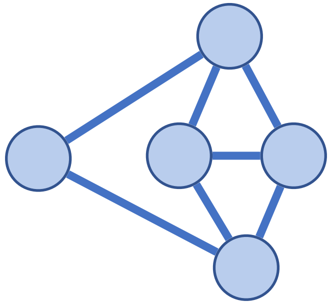
Objective: Visit every vertex and every edge in the graph.

How can we systematically go through a complex graph in the fewest steps?

Tree traversals won't work — lets compare:



- Rooted
- Acyclic
- A clear 'endpoint'



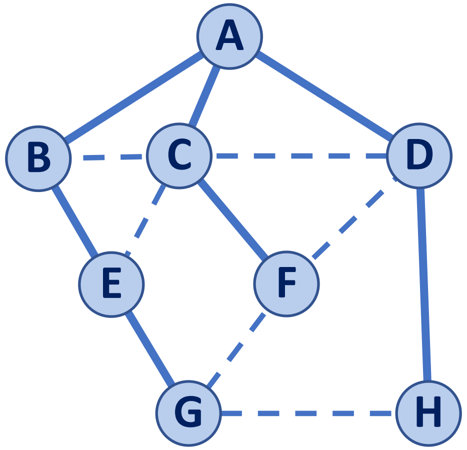
- No root (any start position valid)
- Cycles
- No obvious 'endpoint'

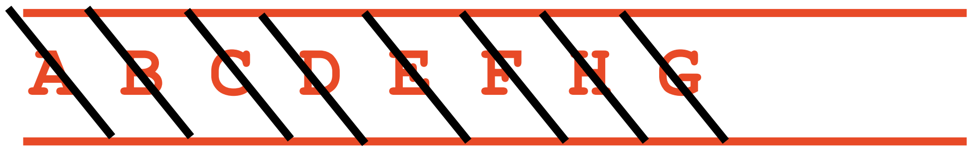
```
BFS(G, v):
  Queue q
  setDist(v, 0)
  q.enqueue(v)

  while !q.empty():
    v = q.dequeue()

    foreach (Vertex w : G.adjacent(v)):
        if( getDist(w) == -1):
            setLabel((v, w), DISCOVERY)
            setPred(w, v)
            setDist(w, v + 1)
            q.enqueue(w)
        else:
            setLabel((v, w), CROSS)
```

v	d	P	Adjacent Edges
A	0	-	B C D
B	1	A	A C E
C	1	A	A B D E F
D	1	A	A C F H
E	2	B	B C G
F	2	C	C D G
G	3	E	E F H
H	2	D	D G





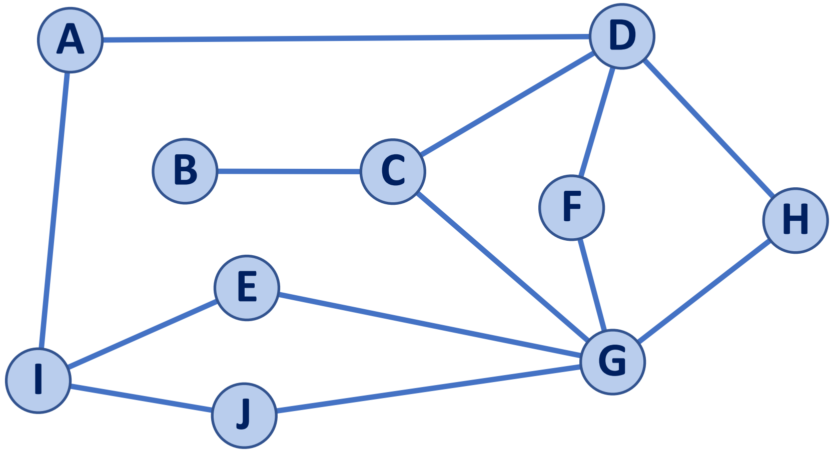
BFS Observations

1. BFS can be used to count components
2. BFS can be used to detect cycles

3. The BFS 'distance' value is always the shortest distance from source to any vertex (and the discovery edges form a MST)

4. The endpoints of a cross edge never differ in distance by more than 1 (**|d(u) - d(v)| = 1**)

Traversal: DFS



Initialize dist / pred / stack

All dist null (start node dist 0)

All pred -1 (start node pred -1)

Stack loaded with start node

While stack not empty

tmp=stack.peek()

Process one child of tmp

Add to stack

dist = tmp.dist+1

pred = tmp

If no unvisited children

stack.pop()

A

Stack

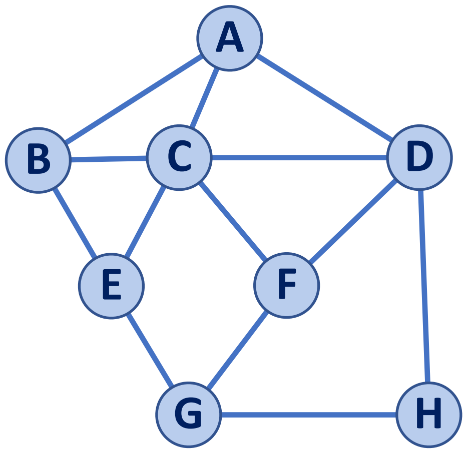
```
DFS(G):
  foreach (Vertex v : G.vertices()):
   setPred(v, NULL)
   setDist(v, -1)

  foreach (Edge e : G.edges()):
    setLabel(e, UNEXPLORED)

  foreach (Vertex v : G.vertices()):
    if getDist(v) == -1:
       DFS(G, v)
```

```
DFS(G, v):

    foreach (Vertex w : G.adjacent(v)):
       if( getDist(w) == -1):
          setLabel((v, w), DISCOVERY)
          setPred(w, v)
          setDist(w, v + 1)
          DFS(G, w)
       else:
          setLabel((v, w), BACK)
```



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    foreach (Vertex w : G.adjacent(v)):
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            setLabel((v, w), DISCOVERY)
            setPred(w, v)
            setDist(w, v + 1)
            DFS(G, w)
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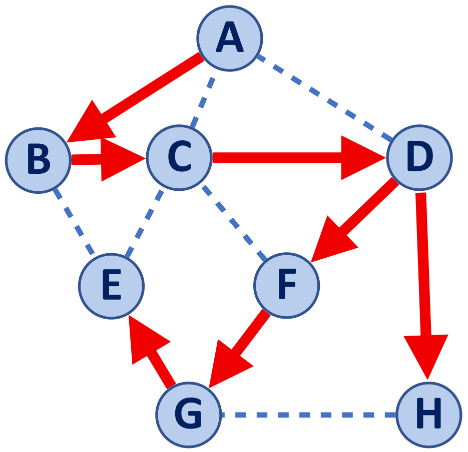


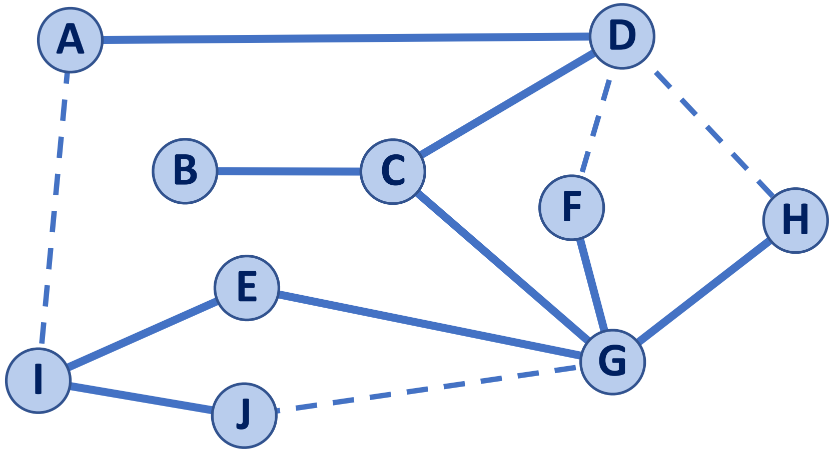
v	d	P	Adjacent Edges
A	0	-	B C D
B	1	A	A C E
C	2	B	A B D E F
D	3	C	A C F H
E	6	G	B C G
F	4	D	C D G
G	5	F	E F H
H	4	D	D G

State of stack when H is added:

H
~~E~~
~~G~~
~~F~~
D
C
B
A

Traversal: DFS

Do we still make a spanning tree?



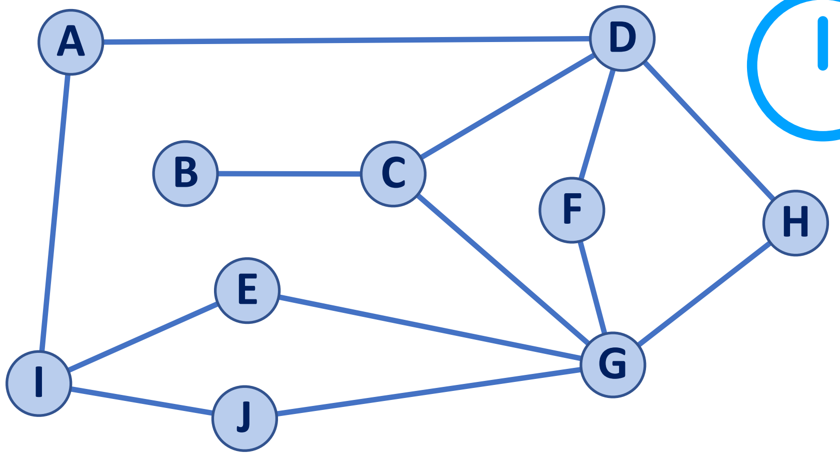
Does distance have meaning here?

Discovery Edge

Do our edge labels have meaning here?

Back Edge

Running time of DFS



Labeling:

- Vertex:

- Edge:

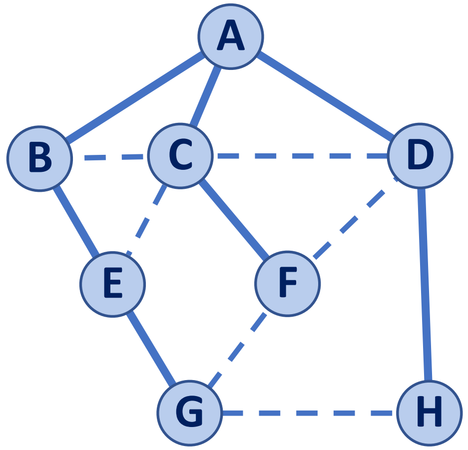
Traversal:

- Vertex:

- Edge:

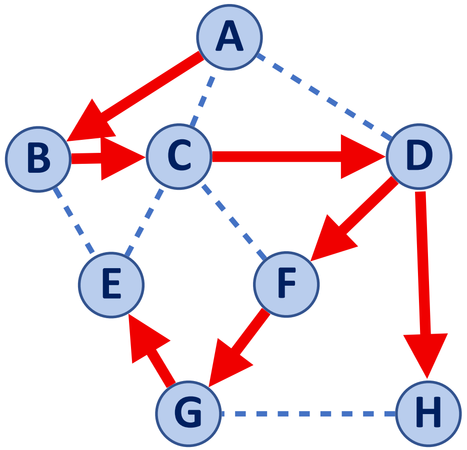
Efficiency: DFS vs BFS

BFS:



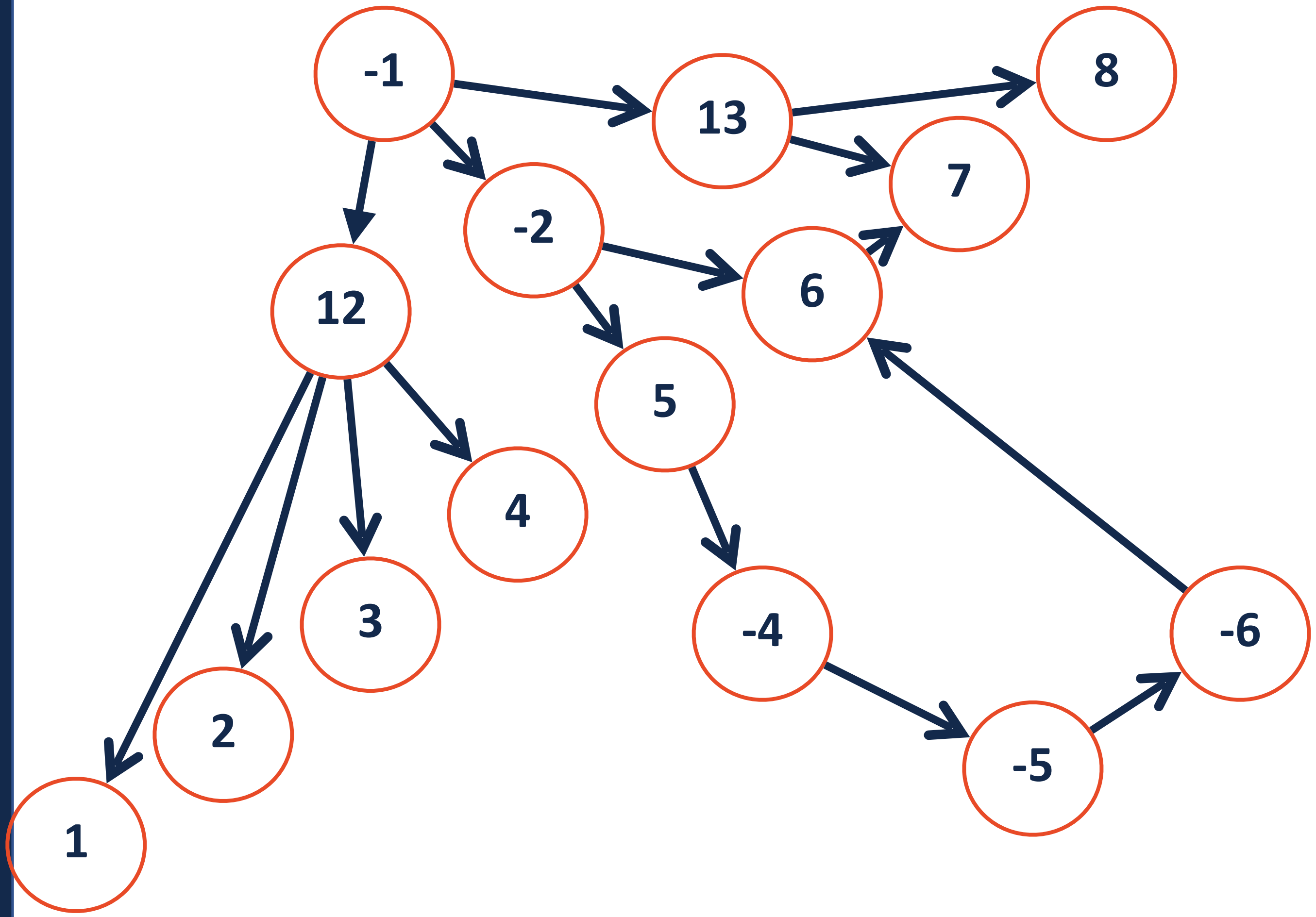
A B C D E F H G

DFS:



A B C D F G E H

Space Efficiency: DFS vs BFS

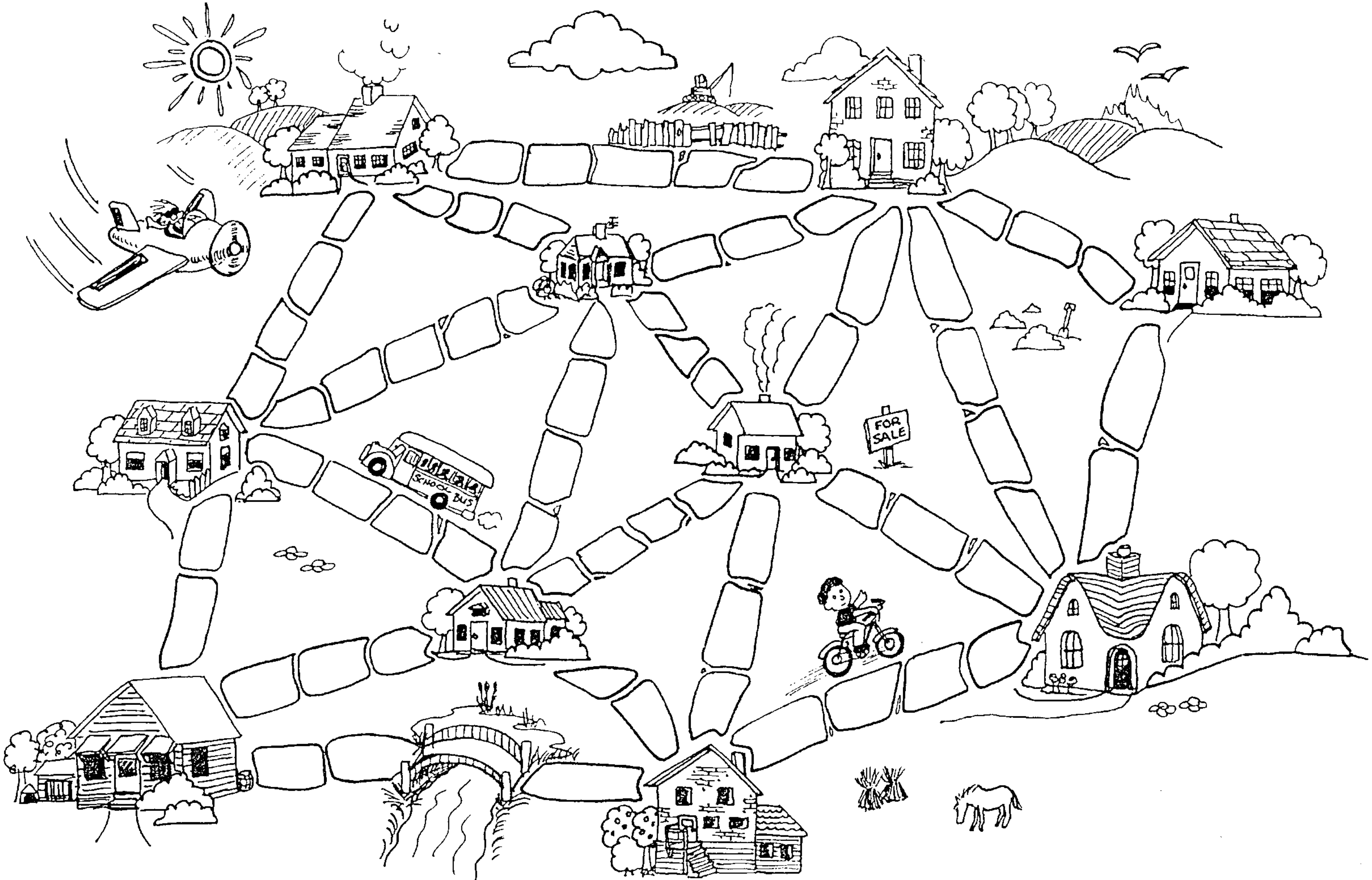


Summary: DFS and BFS

$|V| = n, |E| = m$

Both are **O(n+m)** traversals! They label every edge and every node

BFS	**DFS**
Solves unweighted MST	Solves unweighted MST
Solves shortest path	
Solves cycle detection	Solves cycle detection
Memory bounded by width	Memory bounded by longest path



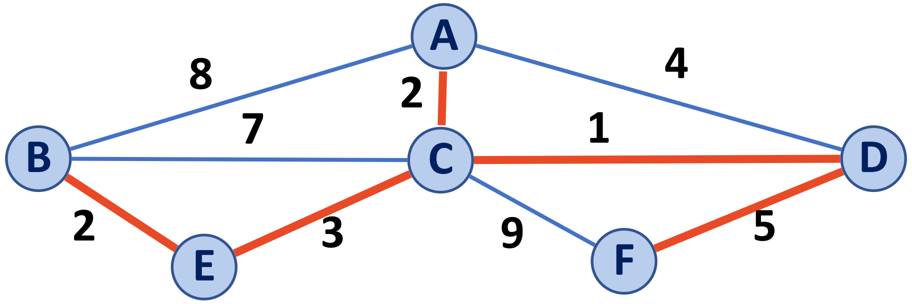
SCHOOL BUS
FOR SALE

Minimum Spanning Tree Algorithms

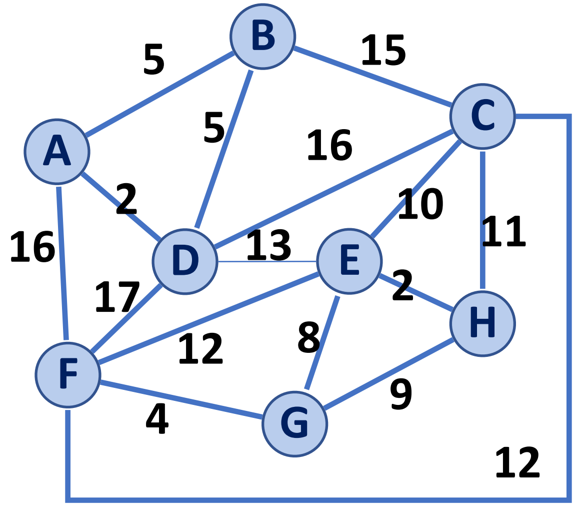
Input: Connected, undirected graph **G** with edge weights (unconstrained, but must be additive)

Output: A graph G’ with the following properties:

- G’ is a spanning graph of G
- G’ is a tree (connected, acyclic)
- G’ has a minimal total weight among all spanning trees



Kruskal's Algorithm *(A graph algorithm for the MST problem)*



What information do I need to get efficiently?

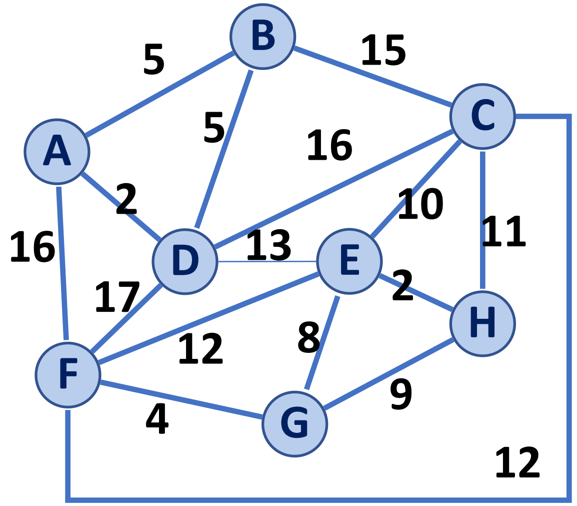
1) The global minimum edge weight

2) Edge connections given a vertex

3) If two vertices are already connected

4) A visited list of vertices

Kruskal's Algorithm *(A graph algorithm for the MST problem)*



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What does this tell me about my implementation choices?

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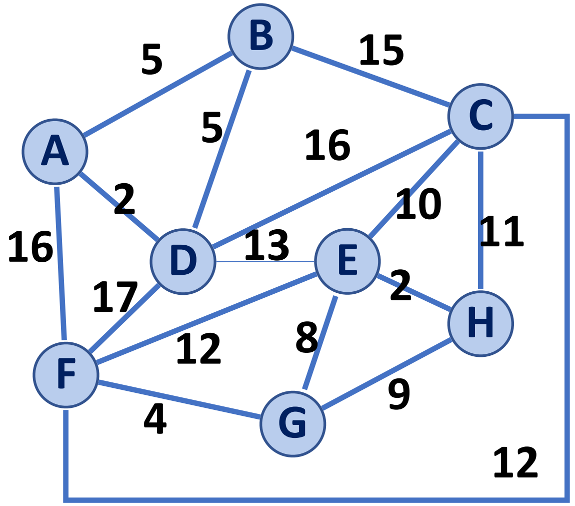


What information do I need to get efficiently?

1) The global minimum edge weight

3) If two vertices are already connected

What does this tell me about my implementation choices?

1) We need a **priority queue** of edges (sorted by weight)

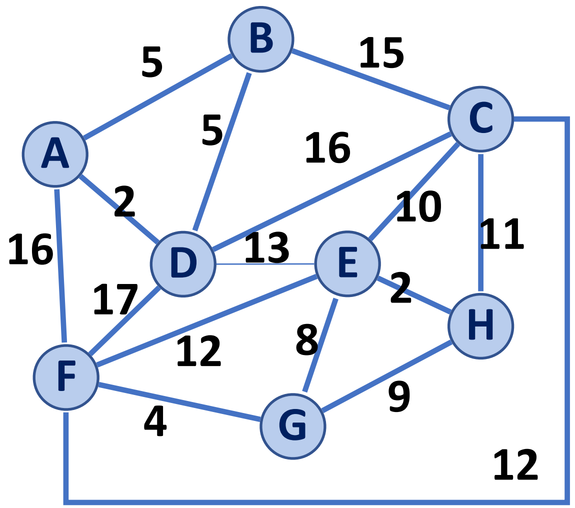
2) We need a **disjoint set** of vertices

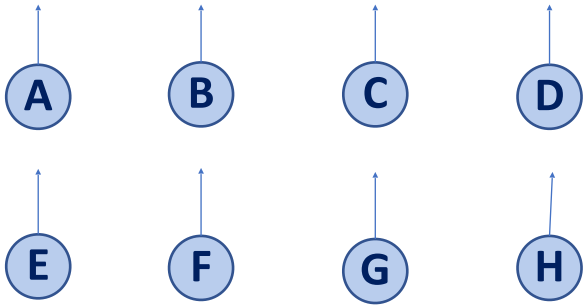
Kruskal's Algorithm

1) Build a **priority queue** on edges

2) Build a **disjoint set** on vertices

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
(G, E)
(G, H)
(E, C)
(C, H)
(E, F)
(F, C)
(D, E)
(B, C)
(C, D)
(A, F)
(D, F)





Kruskal's Algorithm

(A, D)
(E, H)
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(B, C)
(C, D)
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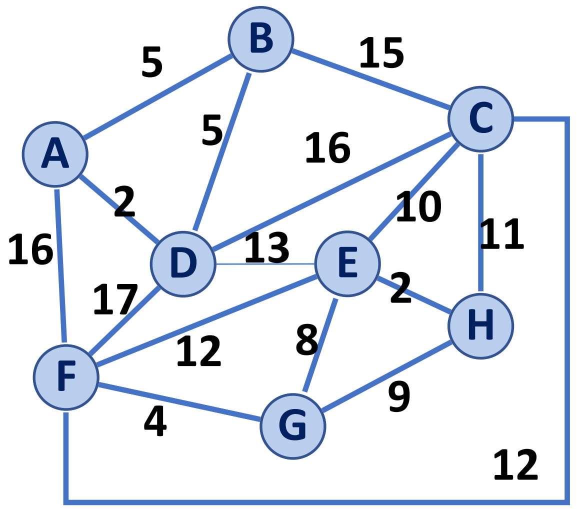
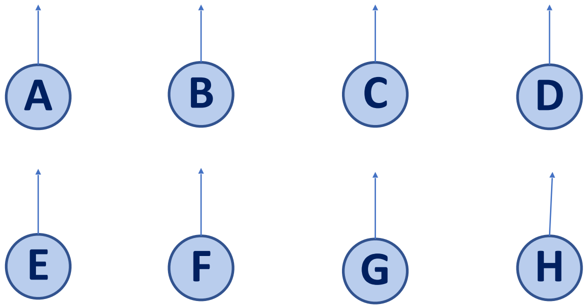
(D, F)





1) Build a **priority queue** on edges

A minheap

or

A sorted array

2) Build a **disjoint set** on vertices

All vertices start as their own set

3) Loop through min edges

If edge connects two disjoint sets

Union sets and record edge in MST

4) Stop when:

N-1 edges recorded

Only a single disjoint set remains

Kruskal's Algorithm

(A, D)
(E, H)
(F, G)
(A, B)
(B, D)
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(G, H)
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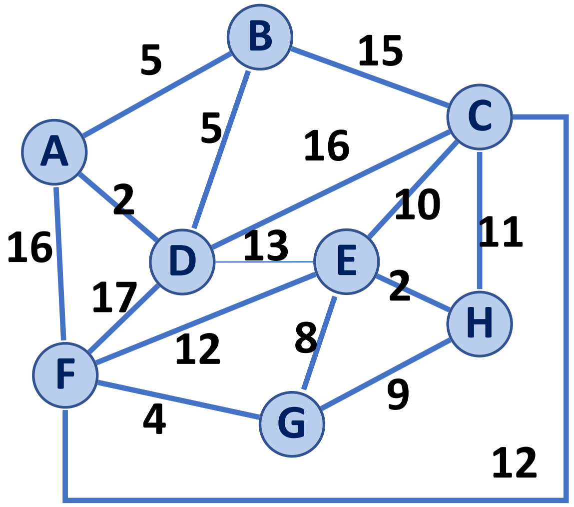

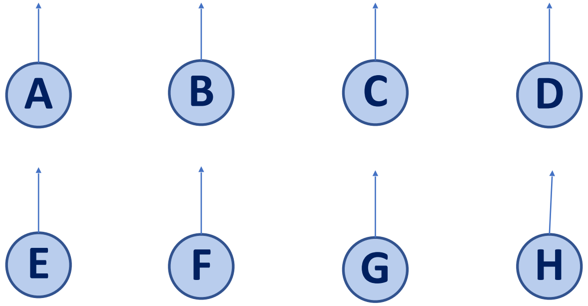


```
KruskalMST(G):
  DisjointSets forest
  foreach (Vertex v : G.vertices()):
    forest.makeSet(v)

  PriorityQueue Q    // min edge weight
  Q.buildFromGraph(G.edges())

  Graph T = (V, {})

  while |T.edges()| < n-1:
    Vertex (u, v) = Q.removeMin()
    if forest.find(u) != forest.find(v):
       T.addEdge(u, v)
       forest.union( forest.find(u),
                     forest.find(v) )

  return T
```

Kruskal's Algorithm

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```

|V|= n,|E|= m

What is the Big O?

Kruskal’s Algorithm

```
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15        forest.union( forest.find(u),
16                      forest.find(v) )
17
18   return T
19
```

$|V| = n, |E| = m$

What is the Big O?

2 — 4: O(n)

6 — 7: Heap: O(m)
Sorted List: O(m log m)

11: m x <12-17>

12—17: Heap: O(log m)
Sorted List: O(1)

Disjoint set we treat as O(1) b/c path compression w/ smart union

Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7		
Each removeMin :12		

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Kruskal's Algorithm

Priority Queue:	Heap	Sorted Array
Building :7	O(m)	O(m log m)
Each removeMin :12	O(m log m)	O(m)

Both result in m + m log m

Why is heap good?

If edge weights can change!

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```

Why is sorted array good?

Sorted array not destroyed and can be useful in other algorithms!

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	
Sorted Array	

```
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  return T
```

Kruskal's Algorithm

Priority Queue:	Total Running Time
Heap	O(n)+O(m) + O(m log m)
Sorted Array	O(n) + O(m log m) + O(m)

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```

Neat trick: O(log m) = O(log n)

Bounds of n & m relationship?

$$n - 1 \leq m \leq n^2$$

Connected graph *Complete graph*