Errors

Scientific Notation

In **scientific notation**, a number can be expressed in the form

$$x = \pm r \times 10^m$$

where r is a coefficient in the range $1 \leq r < 10$ and m is the exponent.

$1165.7 =$

$0.0004728 =$

Error in Numerical Methods

- Every result we compute in Numerical Methods contain errors!
- We always have them… so our job? Reduce the impact of the errors
- How can we model the error?

$$\text{Approximate result} = \text{True Value} + \text{Error}$$

$$\bar{x} = x + \Delta x$$

- **Absolute error:** $|x - \bar{x}|$
- **Relative error:** $\dfrac{|x - \bar{x}|}{|x|}$

- Absolute errors can be misleading, depending on the magnitude of the true value x.

- For example, let's assume an absolute error $\Delta x = 0.1$

  - $x = 10^5 \rightarrow \bar{x} = x + \Delta x = 10^5 + 0.1$ (accurate result)
  - $x = 10^{-5} \rightarrow \bar{x} = x + \Delta x = 10^{-5} + 0.1$ (inaccurate result)

- Relative error is independent of magnitude.

You are tasked with measuring the height of a tree which is known to be exactly 170 ft tall. You later realized that your measurement tools are somewhat faulty, up to a relative error of 10%. What is the maximum measurement for the tree height (numbers rounded to 3 sig figs)?

A) 153 ft

B) 155 ft

C) 187 ft

D) 189 ft

You are tasked with measuring the height of a tree and you get the measurement as 170 ft tall. You later realized that your measurement tools are somewhat faulty, up to a relative error of 10%. What is the minimum height of the tree (numbers rounded to 3 sig figs) ?

A) 153 ft

B) 155 ft

C) 187 ft

D) 189 ft

Significant digits

Significant figures of a number are digits that carry meaningful information. They are digits beginning to the leftmost nonzero digit and ending with the rightmost "correct" digit, including final zeros that are exact.

The number 3.14159 has six significant digits.

The number 0.00035 has two significant digits.

The number 0.000350 has three significant digits.

Accurate to *n* significant digits means that you can trust a total of *n* digits. *Accurate digits* is a measure of relative error.

Relative error: $error = \frac{|x_{exact} - x_{approx}|}{|x_{exact}|} \leq 10^{-n+1}$

n is the number of accurate significant digits

Suppose x is the true value and $\tilde{x}$ the approximation.

$\tilde{x}$ has $\boldsymbol{n}$ **significant figures** of x if $|x - \tilde{x}|$ has zeros in the first n decimal places counting from the leftmost nonzero (leading) digit of x, followed by a digit from 0 to 4.

Example:

$x = 3.141592653$

$\tilde{x} = 3.14159 \longrightarrow |x - \tilde{x}| = \underbrace{0.00000}_{\text{6 zeros}}2653 = 2.653\times10^{-6} \longrightarrow$ $\tilde{x}$ has **6 sf**

$\tilde{x} = 3.1415 \longrightarrow |x - \tilde{x}| = \underbrace{0.0000}_{\text{5 zeros}}92653 = 0.92653\times10^{-4} \longrightarrow$ $\tilde{x}$ has **4 sf**

$\tilde{x} = 3.1416 \longrightarrow |x - \tilde{x}| = \underbrace{0.00000}_{\text{6 zeros}}7347 = 0.7347\times10^{-5} \longrightarrow$ $\tilde{x}$ has **5 sf**

So far, we can observe that $|x - \tilde{x}| \leq 5\times10^{-n}$. Note that the exact number in this example can be written in the scientific notation form $x = q\times10^{0}$. *What happens when the exponent is not zero?*

Relative error:

$$e_r = \frac{|x-\tilde{x}|}{|x|} = \frac{|x-\tilde{x}|}{q\times10^m} \leq \frac{5\times10^{-n}10^m}{q\times10^m} = \frac{5}{q}\times10^{-n} \leq 5\times10^{-n}$$

Or in a more general way, we will use the relation:

$$e_r \leq 10^{-n+1}$$

$$\log_{10}(e_r) \leq 1-n \quad \Rightarrow \quad n \leq 1-\log_{10}(e_r)$$

For example, if relative error is 10^{-2} then $\tilde{x}$ has at most 3 significant figures of x

After rounding, the resulting number has 5 accurate digits. What is the tightest estimate of the upper bound on my relative error?

$A)\ 10^{5}$

$B)\ 10^{-5}$

$C)\ 10^{4}$

$D)\ 10^{-4}$

Sources of Error

Main source of errors in numerical computation:

- **Rounding error:** occurs when digits in a decimal point ($1/3 = 0.3333...$) are lost (0.3333) due to a limit on the memory available for storing one numerical value.
- **Truncation error:** occurs when discrete values are used to approximate a mathematical expression (eg. the approximation $\sin(\theta) \approx \theta$ for small angles θ)