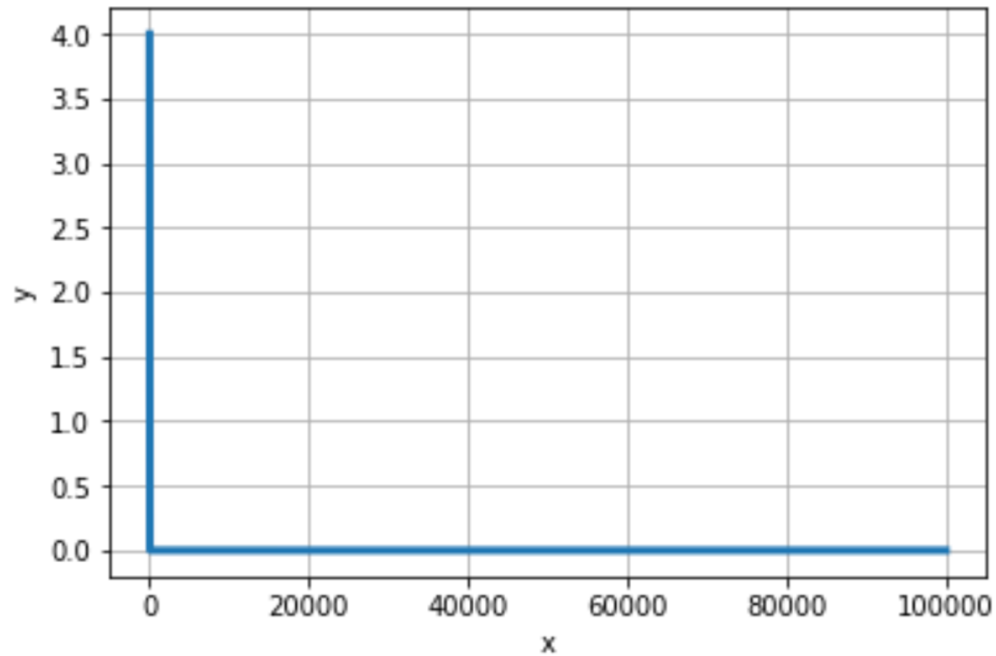
Convergence plots and Big-O notation

Let’s first talk about plots…

- Power functions:

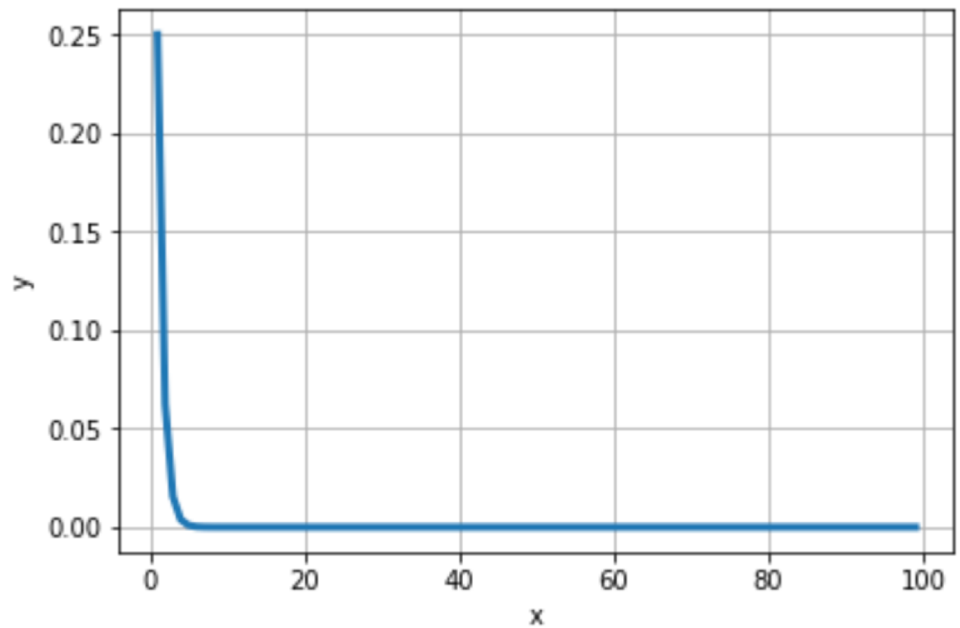
  $$y = a\, x^b$$



Let's first talk about plots…

- Exponential functions:

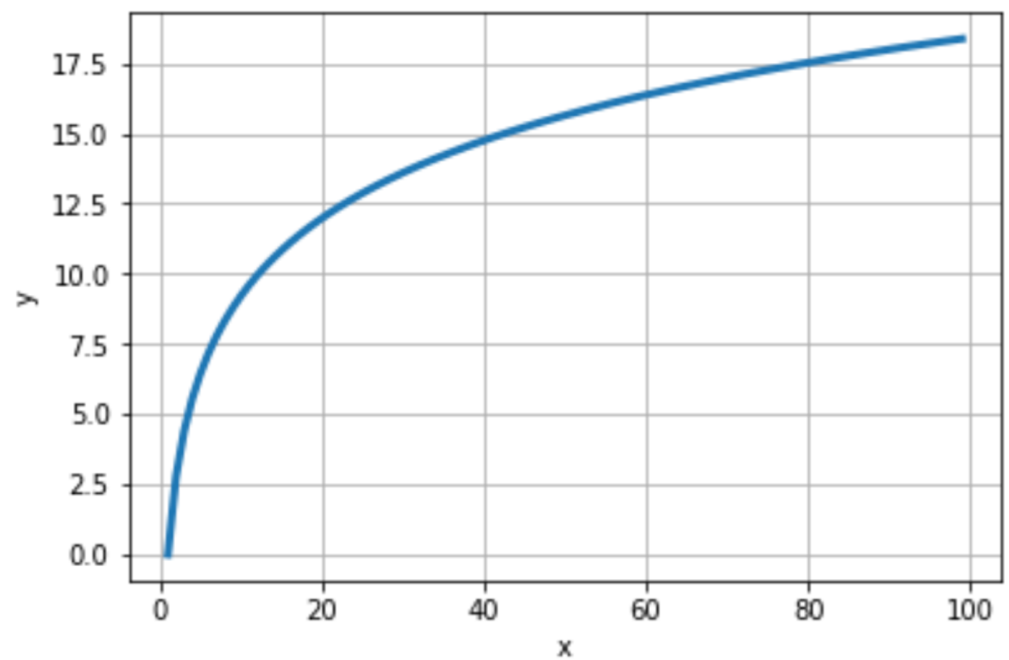
$$y = a\, b^x$$



Let's first talk about plots…

- Log functions:

$$y = a\log(b\,x)$$



Matrix-matrix multiplication example

For a matrix with dimensions $n \times n$, the computational complexity can be represented by a power function:

$$time = c\, n^a$$

We could count the total number of operations to determine the value of the constants above, but instead, we will get an estimate using a numerical experiment where we perform several matrix-matrix multiplications for vary matrix sizes, and store the time to take to perform the operation.

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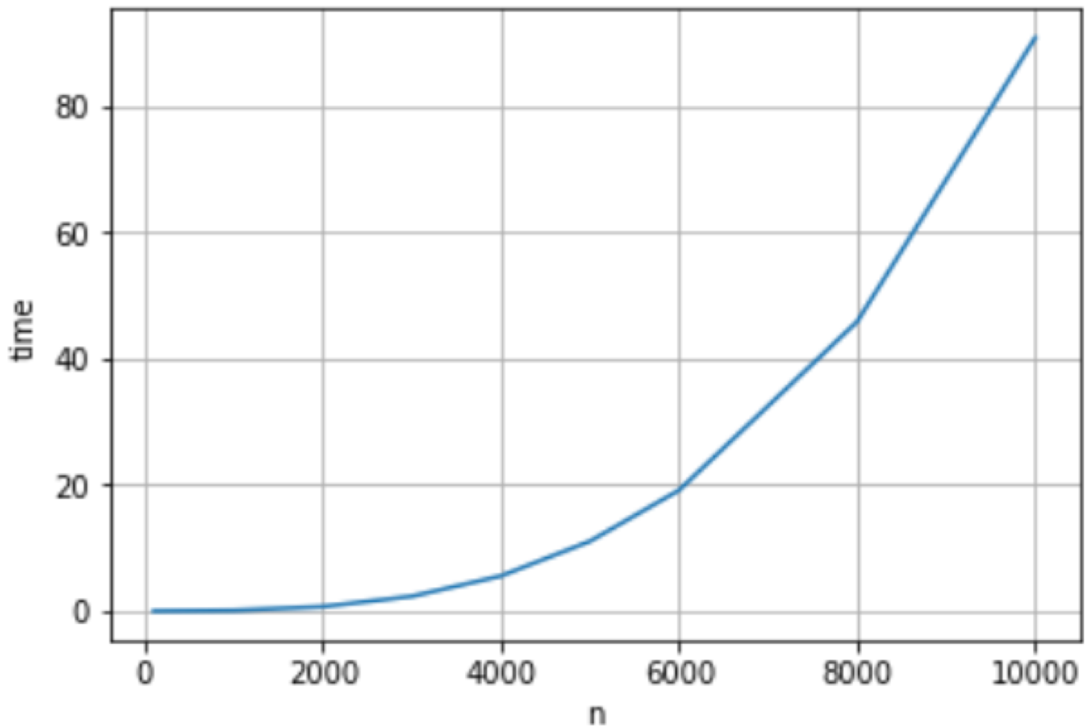
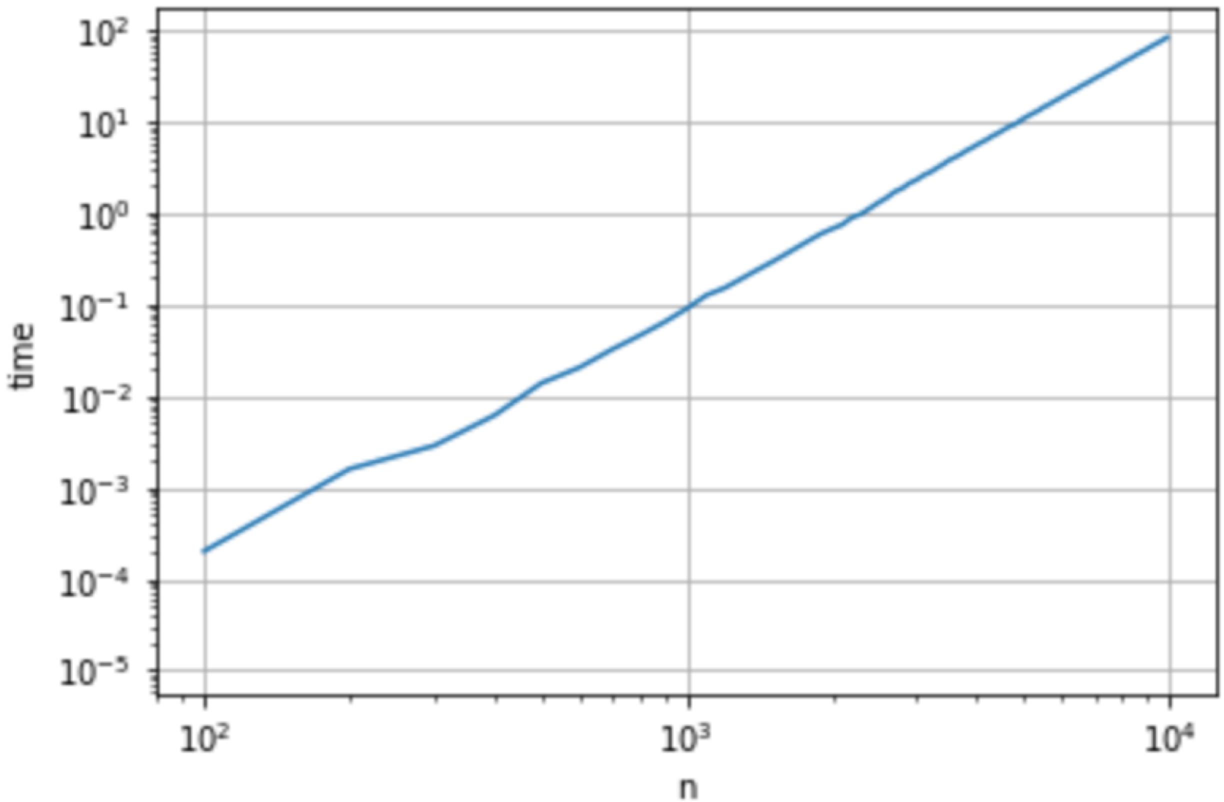


What type of plot will result in a straight line?

A) semilog-x B) semilog-y C)log-log

Power functions are represented by straight lines in a log-log plot, where the coefficient a is determined by the slope of the line.

$$time = c\, n^{a}$$



Demo: Cost of Matrix-Matrix Multiplication

Asymptotic Behavior; (“Big O”) $O(\cdot)$ Notation

How do we say something exact without having to predict individual values exactly?

Let $g(n)$ be our model function.

Revising Big-Oh notation

Let f and g be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to \infty$$

If an only if there is a positive constant M such that for all sufficiently large values of x, the absolute value of $f(x)$ is at most multiplied by the absolute value of $g(x)$. In other words, there exists a value M and some x_0 such that:

$$|f(x)| \le M\,|g(x)| \quad \forall\, x \ge x_0$$

Example:

Consider the function $f(x) = 2x^2 + 27x + 1000$

Revising Big-Oh notation

Let f and g be two functions. Then

$$f(x) = O\big(g(x)\big) \text{ as } x \to a$$

If an only if there exists a value M and some δ such that:

$$|f(x)| \leq M\,|g(x)| \quad \forall x \;\; where\; 0 < |x - a| < \delta$$

Same example…

Consider the function $f(x) = 2x^2 + 27x + 1000$

Iclicker question

Suppose that the truncation error of a numerical method is given by the following function:

$$E(h) = 5h^2 + 3h$$

Which of the following functions are Oh-estimates of $E(h)$ as $h \to 0$

1) $O(5h^2)$
2) $O(h)$
3) $O(5h^2 + 3h)$
4) $O(h^2)$

Mark the correct answer:

A) 1 and 2

B) 2 and 3

C) 2 and 4

D) 3 and 4

E) NOTA

Iclicker question

Suppose that the complexity of a numerical method is given by the following function:

$$c(n) = 5n^2 + 3n$$

Which of the following functions are Oh-estimates of $c(n)$ as $n \to \infty$

1) $O(5n^2 + 3n)$
2) $O(n^2)$
3) $O(n^3)$
4) $O(n)$

Mark the correct answer:

A) 1,2,3

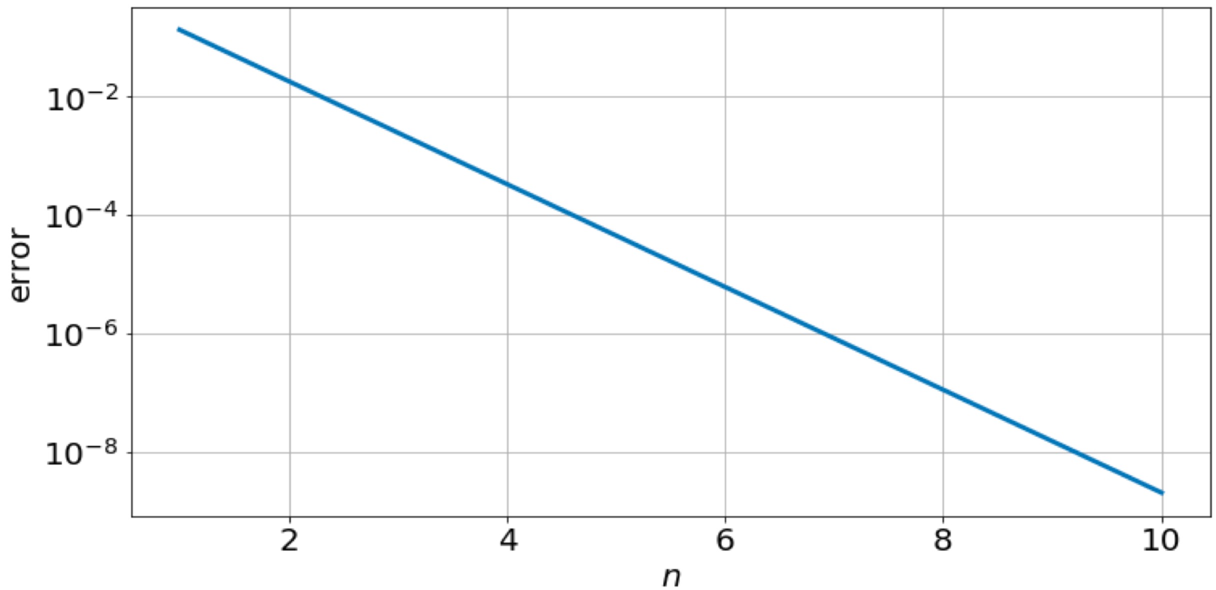
B) 1,2,3,4

C) 4

D) 3

E) NOTA

Select the function that best represents the decay of the error as n increases

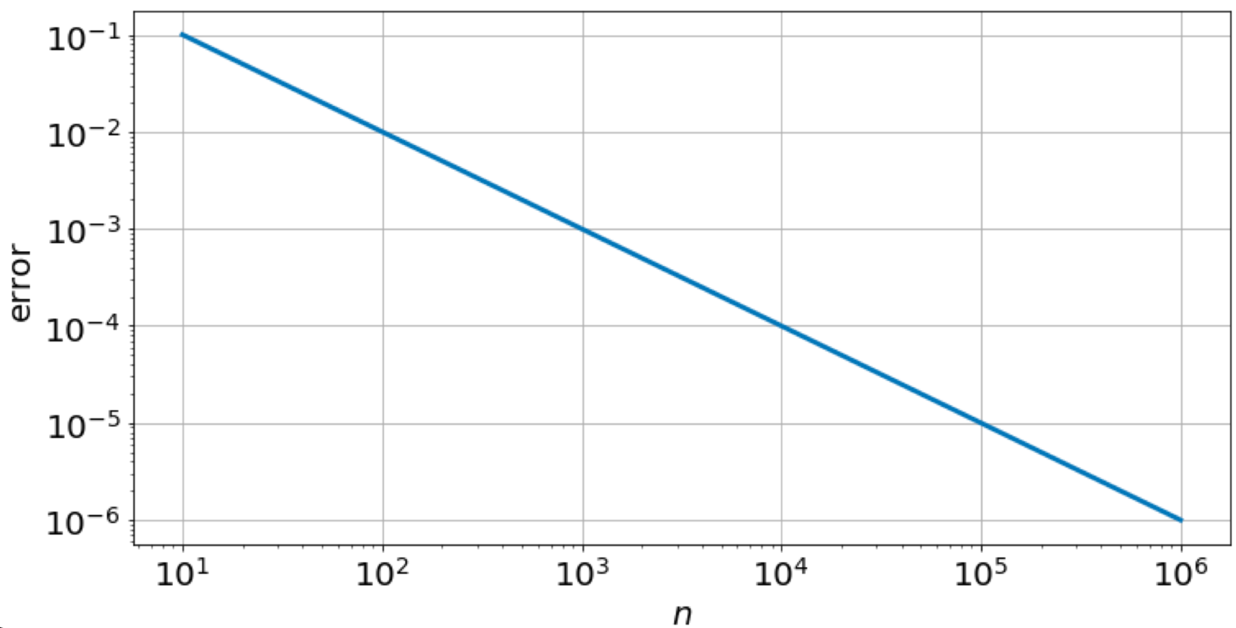


A) e^{-2n}

B) e^{-n}

C) n^{-1}

D) n^{-2}



A) e^{-2n}

B) e^{-n}

C) n^{-1}

D) n^{-2}

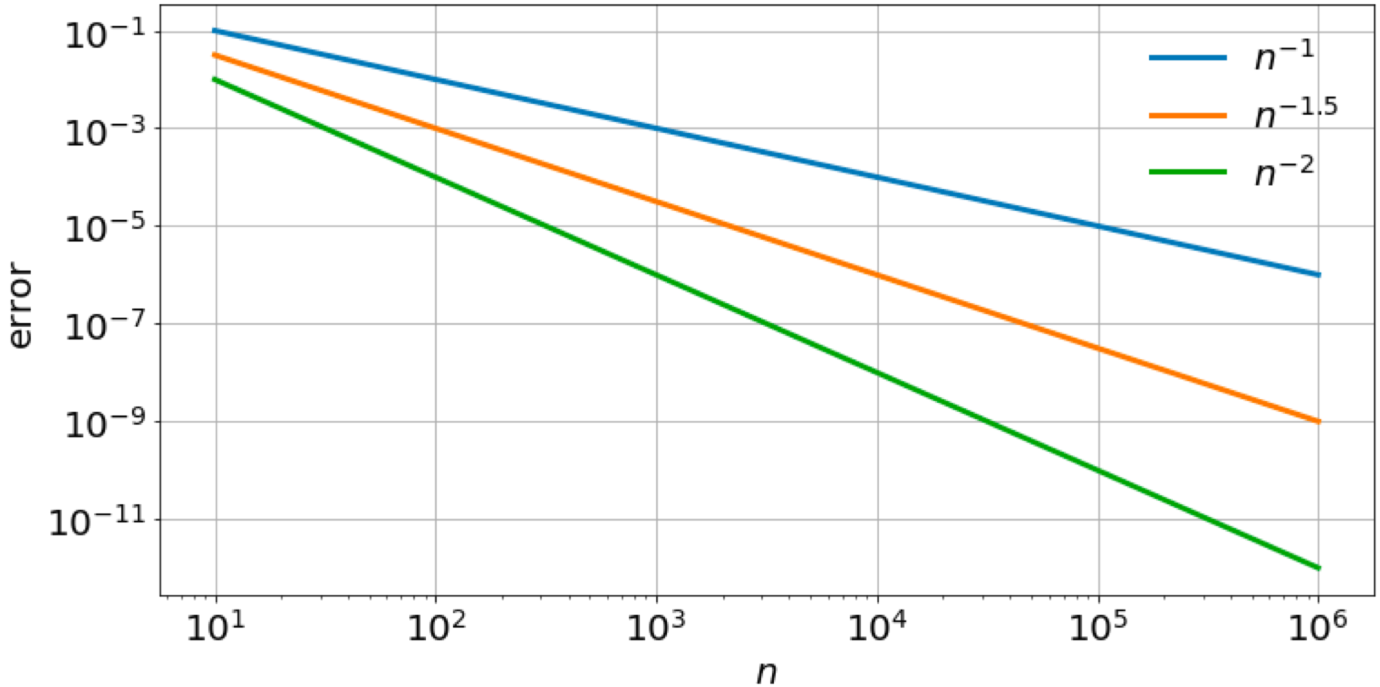
Rates of convergence

1) Algebraic convergence: $error \sim \dfrac{1}{n^{\alpha}}$ or $O\left(\dfrac{1}{n^{\alpha}}\right)$

Algebraic growth: $time \sim n^{\alpha}$ or $O(n^{\alpha})$

α: Algebraic index of convergence

A sequence that grows or converges algebraically is a straight line in a log-log plot.



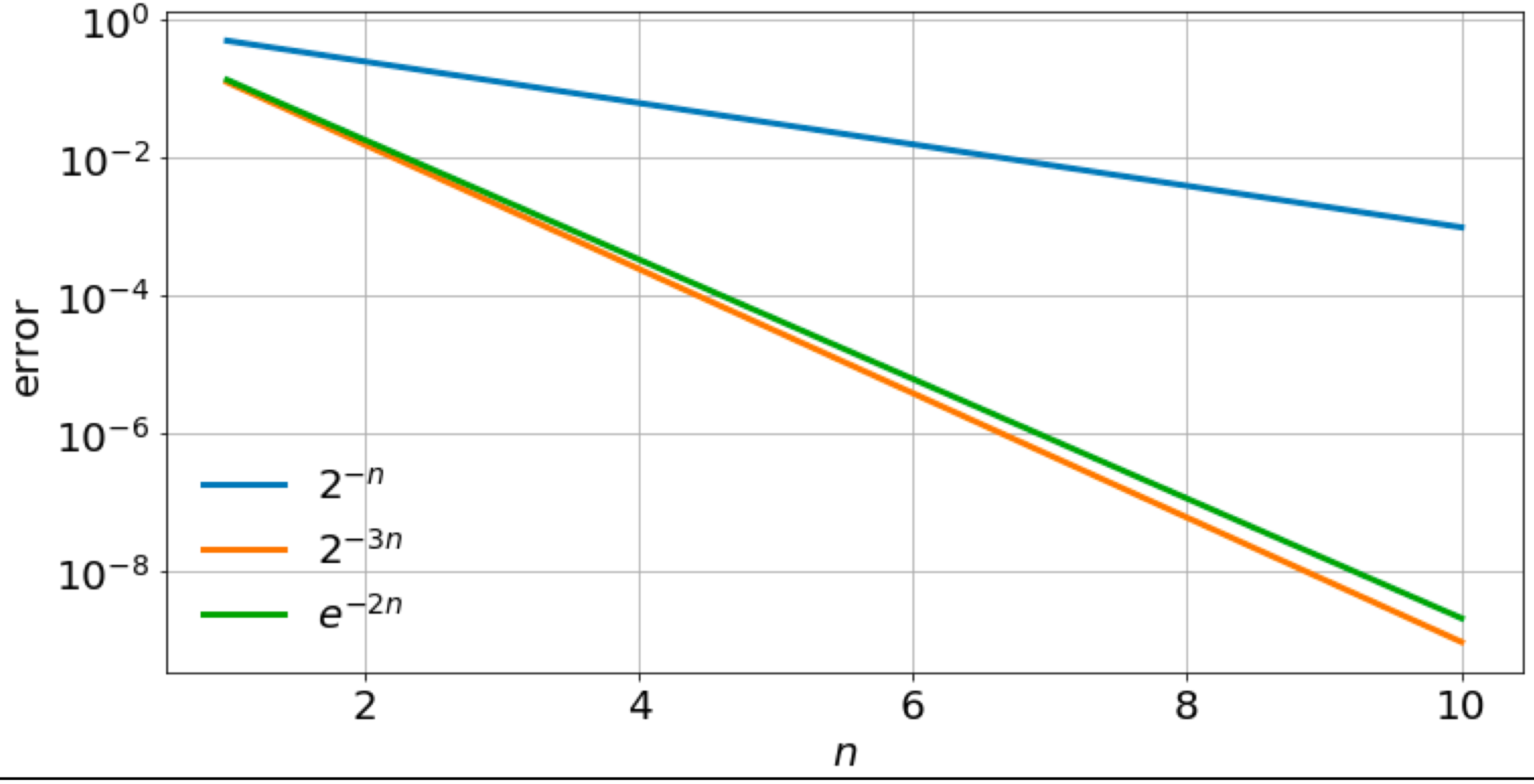
Demo "Exponential, Algebraic and Geometric convergence"

Rates of convergence

2) Exponential convergence: $error \sim e^{-\alpha n}$ or $O(e^{-\alpha n})$

Exponential growth: $time \sim e^{\alpha n}$ or $O(e^{\alpha n})$

A sequence that grows or converges exponentially is a straight line in a linear-log plot.



Rates of convergence

Exponential growth/convergence is much faster than algebraic growth/convergence.